AN

INTRODUCTION

TO

ARITHMETIC

AND

ALGEBRA.

BY THOMAS MANNING.

Vol I

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Cambridge:
PRINTED BY BENJAMIN FLOWER:
AND SOLD BY NICHOLSON, LUNN, AND DEIGHTON, CAMBRIDGE;
COOKE, OXFORD; STEVENSON, NORWICH; RIVINGTONS
AND WINGRAVE, LONDON.

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PREFACE.

ON introducing a work to the public, it may be proper to explain the motives from which it has been undertaken, and the plan upon which it has been conducted. Concerning the former of theſe points I have only to obſerve, that it has been my intention to ſupply an Introduction to Arithmetic and Algebra, in which every propoſition ſhould be not only ſo accurately demonſtrated, but ſo fully explained, as to prevent the *neceſſity* of additional aſſiſtance. And though perhaps by many it may be conceived, that the clear, diffuſe and elaborate Treatiſe of SAUNDERSON,—the explanatory Introduction of LUDLAM—and the late judicious and comprehenſive Syſtem of Mr. WOOD, of St. John's College, have left no want of ſuch a publication; yet without, in any degree, preſuming to rival, or intending to depreciate the various excellencies of theſe, and other admirable productions on the ſame ſubject, I am inclined to imagine that my time has not been altogether miſemployed, in endeavouring to elucidate, by the moſt laborious exactneſs, whatever might ſtill appear difficult or abſtruſe in the ſubject upon which I have treated. And in ſome meaſure to protect myſelf from the cenſure of thoſe who may be inclined to deſpiſe all further attempts to familiarize the rudiments of ſcience, I ſhall avail myſelf of the obſervation and example of an excellent mathematician and diſtinguiſhed ſcholar, whoſe words I ſhall not heſitate to produce on the preſent occaſion: "Levia quidem hæc, & parvi "forte, ſi per ſe ſpectentur, momenti. Sed ex "elementis conſtant, ex principiis oriuntur, omnia: Et ex judicii conſuetudine in rebus *minutis* adhibita, pendet ſæpiſſime etiam in *maximis* "vera atque accurata ſcientia." *

The means by which I have attempted to accompliſh this object, have been by collecting and adopting whatever I have found ſufficiently accurate and explicit, and as far as has been in my power, by elucidating what was obſcure, and ſupplying what was deficient. But that I may not appear, by a declaration ſo general, too ſlightly to paſs over my obligations to former and preſent authors, I ſhall beg leave here to acknowledge the valuable additions which my book has derived from the publications of **Baron Maseres**, a writer no leſs diſtinguiſhed by the profoundneſs of his reaſoning, than by the accuracy and perſpicuity of his method.

---

* Dr. Clarke.

There are many other Treatiſes from which I have derived conſiderable aſſiſtance, and eſpecially, both with reſpect to matter and to the arrangement of it, from the late publication of Mr. **Wood**, of St. John's. It would indeed be impoſſible to enumerate every inſtance in which I have either accidentally or deſignedly been indebted to other writers—though I believe few will be found in which I have adopted any thing without conſiderable additions or alterations.

While purſuing this train of minute and rigid demonſtration, I have occaſionally diſcovered, and I hope, ſometimes, ſucceſsfully ſupplied proofs, which have hitherto been either altogether wanting, or at leaſt defective. And paſſing over others, as of leſs importance, I truſt it may be permitted me here to expreſs my hopes, that the demonſtrations I have given of the Binomial Theorem will not be unacceptable to the reader. That it is very eaſy to eſtabliſh generally the law of the coefficients for any propoſed number of terms, I know from experience; having fallen on ſeveral methods, before I obtained that which is now given. But the difficulty is to make manifeſt the law by which *all* the ſucceeding coefficients muſt neceſſarily be generated; and without which the demonſtration would be only an induction.

It may be thought perhaps by ſome, that in many inſtances I have been unneceſſarily diffuſe in tracing each ſtep of the demonſtration; but it ſhould be recollected, that I am not writing for thoſe, to whom the principles and operations of Algebra are already familiar, nor for ſuch as have the advantage of perſonal inſtruction—But to enable ſtudents, who are not poſſeſſed of either of theſe advantages, to acquire with facility, and without any additional books or aſſiſtance, a complete knowlege of all the ſubjects, contained in this publication. And, however a long and minute demonſtration may convey an appearance of difficulty, yet a little conſideration will make it ſufficiently evident, that in proportion as the ſteps are given at length, the more eaſily and quickly will the demonſtration be underſtood by beginners.

Before I conclude, I cannot neglect to ſtate the high ſenſe I entertain of the obligation conferred upon me, by ſo numerous and reſpectable a ſubſcription. And at the ſame time it may not be improper for me to mention, that whatever may be found in this publication deſerving either of cenſure or commendation, is to be ſolely imputed to myſelf; as it has not been in my power to avail myſelf of the directing information, or the ſuperintending judgment of a ſingle friend or aſſiſtant.

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AS the preſent work has very much exceeded the limits originally expected by the Author, he finds himſelf under the neceſſity of either totally omitting, or of delivering in a future Volume, ſome of the articles which were mentioned in his firſt propoſals, but for which he could not find room in this publication. Reſpecting theſe it is his intention to fulfil his obligations, by immediately ſubmitting them to the public—ſhould their deciſion on the preſent attempt juſtify him in hazarding a ſecond.

Ready for the Preſs, and ſpeedily will be publiſhed,
by Subſcription,

AN
INTRODUCTION
TO
ARITHMETIC
AND
ALGEBRA.

VOL. II.

In which will be explained,

The Proportion of Variable Quantities; the Rule of Three; the principal Properties of Arithmetical and Geometrical Series; the Method of ſolving Geometrical Problems by Algebra; the moſt remarkable Properties of Numbers; the Method of finding the greateſt common Meaſure of Algebraic Expreſſions, and Logarithms.

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ERRATA.

Page 18, line 3 from the bottom, after the words, *product of the* inſert *units figure of the multiplicand multiplied by the* p. 30, l. 13, for 9, read 10 p. 33, l. 7, for 2, read 5 p. 58, l. 16, for *multiplied*, read *multiplicand* p. 62, l. 8, for $a^3by-a^2by^2+aby^3$, read $a^3by-a^2b^2y+ab^3y$ p. 90, l. 6, after $= d. -$. p. 97, l. 3, for $\frac{a}{c}$ read $\frac{a}{b}$ p. 103, l. 5 from the bottom, for 1000 read 10000. p. 119, l. 10, for $\frac{2x^4}{a^4}$ read $\frac{2x^4}{a^2}$ p. 120, for the index $n-1$ read n p. 131, l. 5, for $4a^6b^3$, read $4a^2b^3$ p. 136, l. 4, *d.* , p. 145, l. 3, for a^6, read x^6 p. 146, l. 7, for $2a^2$, read $8a^2$ p. 148, l. 12, for 6^3, read b^3 p. 157, l. 16, for $20aa$, read $200a$ p. 176, laſt line, for *of it*, read *of the ſquare root of it* p. 189, laſt line, for $=2y$, read $-2y$ p. 191, l. 16, for $\frac{8-x-2y}{4}$ read $8-\frac{x-2y}{4}$ p. 193, l. 14, for *quantities*, read *equations* p. 207, l. 5, for 42, read 52. laſt l. for $a-b$, read $a-d$ p. 231, end of l. 4, for $+\sqrt{-3}$, read $-\sqrt{-3}$. p. 235, l. 16, for $\pm\sqrt{60}$. read $1\pm\sqrt{60}$. p. 245, for 1500, read 150000 p. 250 and 251, for 1500, read 150000 254, l. 4 from bottom, for *the article*, read *the* 263d art. p. 256, l. 8, for *two*, read *three*. p. 275, laſt line but one, for 60, read 10. p. 277, l. 5, for *leſs*, read *greater*. p. 284, l. 11, *d.* 3d. p. 293, for the index m, read r.

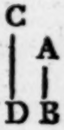
AN

INTRODUCTION
TO
ALGEBRA AND ARITHMETIC.

CHAPTER I.

Article 1. ARITHMETIC is the art of computing the number of things of any ſpecific denomination, in any propoſed caſe: For inſtance, the number of poſitions, with reſpect to order, in which, two, three, or more bodies can be placed.—The number of lines of a certain determined length, that muſt be taken to make up a given line, &c. &c.

2. If from reaſoning founded upon ſome hypotheſis, one thing, as, for example, the line CD, be found compoſed of a certain number of lines AB, e. g. of three of them, CD is called three, if AB be conſidered as one whole. If AB be divided into two equal parts, and CD be found compoſed of a certain number of theſe halves, e. g. of three of them, CD is called three halves; in this caſe, if inſtead of AB, half AB be conſidered



dered as the ſtandard, or one whole, CD is called three, and AB two.

Thus it is that numbers are uſed abſtracted from the name of that, of which they are the numerical exponent; and inſtead of ſaying, that ſuch a number of things added to ſuch a number of things of the ſame kind, forms ſuch a number of things of that kind, we ſay, ſuch a number added to ſuch a number forms ſuch a number: with ſimilar expreſſions in other caſes.

This mode of expreſſion is convenient, becauſe the numerical reſults are not at all affected by the name, of which the numbers are exponents; and, by conſidering the numbers alone without the name, we render the propoſition general and conciſe: Neverthleleſs, any propoſed number muſt be ſuppoſed to ſignify that number of things of a certain kind; for one number abſtractedly bears no relation at all to any other; nor can any thing be truly affirmed concerning the ſum, or difference of two abſtract numbers. Thus, it means nothing to ſpeak of the ſum of two and three, unleſs it be meant to ſpeak of two things and three things of the ſame name added together; Yet 'tis allowable enough to ſay, that two and three make five, becauſe by that expreſſion we only mean to ſignify the generality of the propoſition, that three things added to two things make five things. *

Barrow's Math. Lectures.

---

* With this caution, arithmetic may properly be defined, "The Science of Number."

3. The only changes, of which quantity, or its exponent, number, is ſuſceptible, are augmentation and diminution; conſequently, the only operations in arithmetic are, adding one number to another, and ſubtracting one number from another: Of a few ſmall numbers, the ſum, or remainder that would be left after the ſubtraction of a ſpecified part, is known immediately upon the ſight, or mention of the numbers, becauſe the idea readily ariſes of the effect, which a few ſmall collections of things, or, which a few ſmall collections of things diminiſhed by a ſpecified part, would have upon the ſenſes, and the numerical name of this effect is the anſwer; or elſe, from previous experience the numerical name is aſſociated with the given numbers and conditions; and, converſely, if a ſmall number be propoſed, 'tis readily known how many times it muſt be taken to make up a certain given number, provided that number be not too large; but, not being ſo converſant about large numbers, we cannot promptly name their ſum or difference: † Hence the neceſſity of certain rules founded on the ſyſtem of notation.

---

† A knowledge of the ſums and differences of all numbers whatſoever would ſupercede the uſe of rules for addition, ſubtraction, and mulitplication; and a knowledge of the number of times any propoſed number muſt be taken, in order to make up a given number, would ſupercede the uſe of a rule for diviſion.

4. Numbers are, for convenience of expreſſion and apprehenſion, divided into claſſes, ten of any claſs conſtituting one of the next ſuperior claſs; hence the claſſes are, units, tens, hundreds, thouſands, &c. and there will be nine different numbers belonging to any claſs; in the claſs of units, there will be one unit, two units, three units, &c. to nine units incluſive, or, the numbers one, two, three, &c. to nine incluſive; in the claſs of tens there will be, one ten, two tens, three tens, &c. to nine tens incluſive, or the numbers ten, twenty, thirty, &c. to ninety incluſive; and ſo on. Theſe nine numbers, to whatever claſs they may belong, are reſpectively repreſented by the following nine characters, called ſignificant figures, or digits, viz.

1	2	3	4	5	6	7	8	9
one	two	three	four	five	ſix	ſeven	eight	nine

Any one of theſe characters placed ſingly, denotes the ſimple number here ſubjoined to the character, i. e. ſo many units: When two or more characters are placed together laterally, the firſt, or right hand figure only is to be taken in its ſimple value, the ſecond ſignifies ſo many tens, the third ſo many hundreds, &c. and as it may be required to expreſs a number conſiſting of tens without any units annexed, of hundreds or higher claſſes without all, or any of the inferior claſſes annexed, and as this is to be done by figures ſtanding in their proper reſpective places, while ſome, or all of the lower places have none to fill them up, therefore an additional character (o) cypher, is neceſſary, which ſhall have no ſignification when placed by itſelf, but ſhall ſerve to occupy the vacant places, and bring each ſignificant figure to its proper ſtation.

5. As every number, then, muſt conſiſt of ſo many of ſuch a claſs; or of ſo many of ſuch a claſs, together with ſo many of another inferior claſs, &c. any given number is expreſſed by placing the figure expreſſive of how many it contains of the higheſt claſs to the left hand; next to that the figure expreſſive of how many it contains beſides of the claſs immediately inferior, or, (if it have none of that claſs) a cypher, and ſo on.*

6. For the more eaſy reading of large numbers, they are divided into periods and half periods; each half period conſiſting of three places of figures: The name of the firſt period is units; of the ſecond, millions; of the third, fourth, fifth, ſixth, ſeventh, eighth, ninth, tenth, &c. the names are, reſpectively, billions, trillions, quatrillions, quintrillions, ſextilions, ſeptilions, octilions,

---

* Numbers being in their enunciation divided into claſſes conformably to the above ſyſtem, no difficulty occurs in determining the proper figures and ſtations by which any propoſed number is to be expreſſed.

nonilions, &c. alſo, the right hand half of any period is the units of it; the other half the thouſands.

* The following table, to the number 8437982564738972645, ſhews the names and diviſions of the periods and claſſes as far as trillions, and contains a ſummary of the doctrine of notation.

Suppoſe that the method of expreſſing numbers conſiſting of ſo many of ſuch a claſs together with ſo many of an inferior claſs, &c. had been, by dividing a figure into compartments, and placing the units in the firſt compartment to the right, the tens in the ſecond, and ſo on: Then, to take an example, the number four hundred and thirty-two would have been expreſſed thus $\begin{array}{|c|c|c|}\hline 4&3&2\\\hline\end{array}$; two hundred and ſeven thus $\begin{array}{|c|c|c|}\hline 2&&7\\\hline\end{array}$; eleven thouſand and ninety thus $\begin{array}{|c|c|c|c|c|}\hline 1&1&&9&\\\hline\end{array}$; Take away the diagrams, and fill up the vacant places with cyphers, and the value of each figure will be aſcertained by its poſition the ſame as before, and the notation becomes that in common uſe.

Billions.

		Billions.						Millions.						Units.					
		Thouſands			Units			Thouſands			Units			Thouſands			Units		
&c.	8	4	3	7	9	8	2	5	6	4	7	3	8	9	7	2	6	4	5
&c.	Trillions	Hundreds of Thouſands of Billions	Tens of Thouſands of Billions	Thouſands of Billions	Hundreds of Billions	Tens of Billions	Billions	Hundreds of Thouſands of Millions	Tens of Thouſands of Millions	Thouſands of Millions	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thouſands	Tens of Thouſands	Thouſands	Hundreds	Tens	Units

7. In this ſyſtem it is evident that the value of any figure is increaſed tenfold by its being moved a place higher to the left; thus 7 by itſelf ſignifies ſimply ſeven; but the 7 in 70 or 73, &c. ſignifies ſeven tens; in 700, 711, &c. it ſignifies ſeven hundreds, i. e. ſeven times ten tens, and ſo on; and the ſame is true of any number conſiſting of more than one place, for each of its parts being increaſed tenfold, the whole will be increaſed ten fold; thus 24 by itſelf ſignifies twenty-four, but the 24 in 240 is ten times as great, the 4 units becoming 4 tens, and the 2 tens, 2 hundreds: and converſely the value of any figure or number is decreaſed tenfold, by its being removed a place

lower to the right. A ſimilar ſyſtem of notation might be equally well applied, whatever were the number in any claſs conſtituting one of the next ſuperior claſs; e. g. if that number inſtead of ten had been eight, then, (by a ſimilar method) eight would have been writteu 10; nine, 11; &c. twice eight, 20; twice eight together with one 21; &c. eight times eight, 100; &c. &c. and the value of any figure would be increaſed eightfold by being removed one place to the left.

Again, ſuppoſe the number had been eleven; then there would have been ten numbers in each claſs, and conſequently, there muſt have been ten ſignificant figures: Let ∂ be the additional character, repreſenting ten; then eleven would have been written 10; twelve, 11; eleven together with ten, 1∂; twice eleven, 20; thrice eleven, 30; &c. and the value of any figure would be increaſed eleven fold, by being removed one place to the left. In general, whatever were the number in a claſs conſtituting one of the next ſuperior claſs, that nnmber would denominate how many fold the value of any figure would be increaſed by being removed one place to the left.

ADDITION.

8. ADDITION is that operation by which we find the amount of any two or more numbers.

9. The ſum of any numbers is the ſum of all the parts of thoſe numbers; and it evidently makes no difference in what order the parts be added together: any one part may be taken firſt, and to that, any other part added; to their ſum, a third part, and ſo on; 'till every part of the given numbers have been taken in; and the final reſult will be the true ſum. The moſt convenient method of finding the ſum of any numbers, is by firſt finding the ſum of the units, then, of the tens, and ſo on; if the ſum of the units be leſs than ten, the figure repreſenting that ſum is evidently the units figure of the anſwer; ſince the other parts of the given numbers being tens, hundreds, or ſuperoir claſſes, can never affect the value of the units place; and for the ſame reaſon, if the ſum of the units be equal to, or exceed ten, or tens, the *exceſs* of units muſt be the units figure of the anſwer.

In reckoning up the tens, their ſum muſt be augmented by as many tens, as were found in the ſum of the units: if their ſum ſo augmented be not ſo great as one hundred (i. e. if it do not exceed 9 tens) the digit denominating how many tens there are, muſt be put down as the ſecond figure of the required ſum: but if their ſum be equal to, or greater than an hundred, or hundreds, the exceſs only of tens can be put down in the ſecond place, and the hundreds muſt be carried forward to the third, or hundred row; &c. &c. Thus, for example, if the ſum of 379 and 723 be required, firſt the 9 and 3 units are to be added together; their ſum is 12, or one ten together with 2 units; therefore 2 is put down in the units place of the anſwer; then, this one ten, together with 2 tens and 7 tens exactly making up one hundred, with no exceſs of tens, a cypher is to be put down in the ſecond place; then the one hundred, together with 3 hundreds, and 7 hundreds, making up 11 hundreds, or 1000 together with 100, 1 is to be put down in the third place, and 1 alſo in the fourth place, and the whole ſum is found to be 1102. It is evident, that in this and all other examples, each row may be ſummed as units, as many units being added, as there were found tens in the ſum of the preceding row: Hence the following

RULE FOR ADDITION OF NUMBERS.

10. Place the numbers under each other, units under units, tens under tens, &c. and draw a line

under them; add together the figures in the row of the units, and find how many tens are contained in their ſum; mark down what remains above the tens, or, if nothing remain, a cypher, and carry as many units to the next row as there were tens in the ſum; add up the next row (together with the number carried) in the ſame manner; proceed thus till every row has been ſummed, marking down the full ſum of the laſt row.

EXAMPLES.

Ex. 1.

$$\begin{array}{r}703\\209\\\hline 912\\\hline\end{array}$$

Ex. 2.

$$\begin{array}{r}120\\90\\30\\\hline 240\\\hline\end{array}$$

Ex. 3.

$$\begin{array}{r}7123\\2456\\421\\\hline 10000\\\hline\end{array}$$

Ex. 4.

$$\begin{array}{r}649\\129\\239\\349\\459\\569\\679\\989\\719\\709\\539\\427\\\hline 6456\\\hline\end{array}$$

SUBTRACTION.

11. SUBTRACTION is that operation by which we take a leſs number from a greater, and find their difference, or the exceſs of the greater above the leſs.

12. The number to be ſubtracted, is called the ſubtrahend, the number from which it is to be ſubtracted, the minuend.

The moſt convenient method is to ſubtract like claſſes from like, i. e. units from units; tens from tens, and ſo on; becauſe we have then only to conſider the ſimple difference in the value of the figures; this difference, placed in the ſame claſs (in the anſwer) as the figures were, from which it was obtained, ſhews the true difference of thoſe two figures of the given numbers. For example, if it be required to diminiſh 753 by 621, from the 3 we ſubtract the 1, which leaves 2 for the units place of the anſwer; from the 50 we ſubtract the 20 which leaves 30; therefore 3 is the figure in the ten's place of the anſwer; but the ſimple difference of 2 and 5 gives the ſame ſignificant figure (3): Again, we ſubtract the 600 from the 700, which leaves 100; therefore 1 muſt be put in the third place of the anſwer; which would alſo have been obtained by ſubtracting the ſimple 6 from 7.

It may happen, that the leſs number ſhall have in one or more of its inferior places, a figure of greater value than the correſponding figure of the greater number; the method which ſuggeſts itſelf in this caſe is to detach, or ſuppoſe detached, one from the next ſuperior claſs of the minuend, which being joined to the deficient figure, adds ten to its value; and then the figure of the ſubtrahend can be ſubtracted; but in ſubtracting the next figure, we muſt take care to conſider the figure of the minuend as diminiſhed by one: for example; if it be required to ſubtract 172 from 354, we may conſider the 354 as 200, 150 and 4; and ſubtract the 2 from the 4, the 70 from the 150, (or 7 from 15, the difference being put in the ſecond place of the anſwer) and the 100 from the 200, (or 1 from 2, the difference being put in the third place of the anſwer.)

But it may happen, that a cypher ſtands in the next ſuperior place of the minuend: in that caſe we muſt detach one from the firſt ſuperior claſs that have a ſignificant figure, and diſtribute it among the places that have cyphers; for example, let it be required to ſubtract 2321 from 5000, 5000 is 4000 together with 10 hundreds, or 4000, 900, and 10 tens, or 4000, 900, 90 and 10; under which laſt form the 2321 can be ſubtracted from it, viz. the 1 from the ten, the 20 from the 90, the 300 from the 900, and the 2000 from the 4000; the firſt difference is 9, the ſecond 70, the third 600, and the fourth 2000; therefore the anſwer is 2679.

If, inſtead of conſidering it thus, we add ten to the deficient figure of the minuend, and one to the next figure of the ſubtrahend, we obtain the ſame reſult with more expedition. By this latter method, we (in ſuppoſition) add equally to both minuend and ſubtrahend, and therefore keep their difference unchanged: Hence we are led to the following

RULE FOR SUBTRACTION OF NUMBERS.

13. Place the ſubtrahend under the minuend, units under units, tens under tens, and ſo on; ſubtract units from units, tens from tens, &c. (conſidering the digits in their ſimple value) and place each difference under the figures from which it was obtained. If any figure of the minuend be leſs in value than the correſponding figure of the ſubtrahend, add ten to it, and one to the next figure of the ſubtrahend.

EXAMPLES.

235	403	6340	4000
122	294	6339	3226
113	109	1	774

MULTIPLICATION.

14. MULTIPLICATION is that operation, by which we determine the amount of a number when reckoned twice, thrice, or oftner.

15. The number to be reckoned is called the multiplicand; the number that denominates how often it is to be reckoned, is called the multiplier, and the amount is called the product: Both multiplier and multiplicand are alſo called factors.

This operation then being nothing elſe but the addition of numbers that are equal to one another, if we mark down the given number, whatever it be, the propoſed number of times, and reckon up the ſum, by the rule of addition, we obtain the product: But as this kind of addition is of frequent and extenſive uſe, in order to ſhorten the operation, we mark down the number only once, and by a knowledge of the product of any two ſimple digits, can readily determine the required product. To explain this we muſt obſerve,

Firſt, That any number taken ſo many times, is the ſame as if each part of the number were taken ſo many times, and the products added together: or, dividing the multiplier alſo into parts, the ſame as if each part of the number were taken part of thoſe times, and then each part taken another part of thoſe times, and ſo on, till every part of the number have been taken the whole propoſed number of times; thus, for example, 37 multiplied by 24, or 37 taken 24 times, is the ſame as 7 taken 24 times, together with 30 taken 24 times; alſo, it is the ſame as 7 taken 4 times, together with 30 taken 4 times, together with 7 taken 20 times, together with 30 taken 20 times.

Secondly, That the product of any numbers will conſiſt of the ſame figures, whether thoſe numbers be conſidered ſimply, or whether they repreſent ſo many tens, hundreds, or higher claſſes; the only difference being, that the figures muſt be placed in higher claſſes, according as the factors are in higher claſſes; e. g. if 7 multiplied by 3 produces 21, 70 multiplied by 3 produces the ſame number 21, but in the claſs of tens; i. e. 70 multiplied by 3 is 210; becauſe 70 being ten times as great as 7, 70 taken 3 times in ten times as great as 7 taken 3 times; and for the like reaſon 7 taken 30 times is 210; and 70 taken 30 times (ten times greater than 7 taken 30 times) is 2100; 700 taken 30 times is 21000; &c. and a ſimilar reaſoning obtains, if, inſtead of ſimple digits, we take a number conſiſting of more than one figure; e. g. becauſe 15 multiplied by 42 is 630; 150 multiplied by 42 is 6300; 150 multiplied by 420 is 63000; &c. &c.

16. In general, if we define that to be the claſs of a number, in which its loweſt figure ſtands, the

product of two numbers ſtanding in any claſs is the product of the ſimple numbers, with as many cyphers prefixed to it as muſt be prefixed to the factors, in order to bring them to thoſe claſſes.

17. From theſe conſiderations it follows, that if we multiply each figure of the multiplicand by each figure of the multiplier, and ſet each particular product in the claſs, to which the preceding article determines it, and add the products (ſo placed) together, we ſhall obtain the amount of the whole multiplicand taken as often as there are units in the multiplier.

18. When the ſimple product of the digits conſiſts of more than one place, i. e. is greater than 9, which generally happens, the moſt expeditious method is, to ſet down only the units part of that product; the tens may be added, as units, to the product ariſing from the next higher figure of the multiplicand multiplied by the ſame figure of the multiplier; becauſe this next product, in its true place, is one claſs higher than the prior product; and for a ſimilar reaſon, the units part of this product, ſo augmented, may be ſet down to the left of the units part of the prior product; and, if there be more figures of the multiplicand, the tens may again be carried forward; and ſo on, till we come to the laſt figure of the multiplicand, the whole of the product ariſing from which, (augmented by the tens from the preceding) may be placed to the left of the figures already obtained:

ed: thus but one number will be formed of all the products ariſing from one figure of the multiplier.

If, for example, we have to multiply 47 by 23, according to this method, we firſt multiply 7 by 3; of whoſe product (21) we put down the units part (1); then the 4 by 3; whoſe product (12) placed in the claſs of tens becomes 120, the product of 40 multiplied by 3; to this adding the 20 of the firſt product, it becomes 140; or, to the ſimple 12 adding 2 it becomes 14, which placed to the left of the 1 exhibits 141, the product of 47 multiplied by 3. The product of 47 multiplied by 2 (found after the ſame manner) is 94; therefore the product of 47 multiplied by 20 is 940; if, therefore, in ſetting down this product (94), we place the firſt figure (4) under the tens of the former product, and the next figure (9) under the hundreds, the two numbers will ſtand thus, $\begin{array}{r}141\\94\end{array}$ which added up as they ſtand, give the ſum of 141 and 940, which ſum is the product required. Hence is inferred the following

RULE FOR MULTIPLICATION OF NUMBERS.

19. Place the multiplier under the multiplicand, units under units, tens under tens, and ſo on; find the product of the units figure of the multiplier; ſet down the units part of this product under the units place of the multiplier: Then

find the product of the ſecond figure of the multiplicand multiplied by the units figure of the multiplier; add to it as many units as there were tens in the firſt product; ſet down the units part of this product ſo increaſed to the left of the figure laſt put down, and carry on the tens as before: proceed thus for every figure of the multiplicand, uſing only the units figure of the multiplier, and ſetting down the whole product of the laſt figure of the multiplicand, augmented by the tens carried from the preceding product: If the multiplier conſiſt of more than one figure, repeat the ſame operation with each of the remaining figures ſeparately, in order; obſerving to place the firſt figure of each line under that figure of the multiplier, which is employed in finding that line: collect the ſum of theſe numbers as they ſtand: that ſum will be the product required.

Note. If a cypher or cyphers occupy the right hand place or places of the multiplier or multiplicand, begin the multiplication with the firſt ſignificant figures, prefixing the ſame number of cyphers to the right of the firſt found figure; and if a cypher occur in any other place of the multiplier, paſs it over, obſerving to place the firſt figure of each line under the multiplying digit.

When the digit of the multiplier to be employed is 1, it is uſual, in order to reduce the operation under the general name, to term the ſimple ſetting down of the multiplicand (which takes place in that caſe) " multiplication by 1."

EXAMPLES.

1. Mul^{d} 2984
Mul^{r} 342
5968
11936
8952
Prod. 1020528

2. Mul^{d} 4037069
Mul^{r} 20010
4037069
8074138
Prod. 80781750690

3. Mul^{d} 426000
Mul^{r} 22000
852
852
Prod. 9372000000

4. Mul^{d} 2503
Mul^{r} 12 *
Prod. 30036

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* If a table be formed exhibiting the product of the nine digits multiplied by every number within certain limits, a product may be ſet down in one line, whenever the multiplier does not exceed thoſe limits.

The following Table exhibits the product of any two numbers not greater than 12.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

The firſt column conſiſts of the firſt twelve numbers placed in order; the ſecond column conſiſts of the reſpective products of theſe twelve numbers, when multiplied by 2; the third column conſiſts of their reſpective products when multiplied by 3; and in the ſame manner the ſucceeding columns conſiſt of their products, when multiplied by the ſucceeding numbers reſpectively.

DIVISION.

20. DIVISION is that operation by which we determine how often one number is contained in another.

21. The number contained is called the diviſor; the number containing, the dividend; and the reſult, or number denominating how often the diviſor is contained in the dividend, is called the quotient.

22. Diviſion may be conſidered as the reverſe of multiplication; for as multiplication by 2 doubles any number; by 3 triples it; and ſo on: ſo, diviſion of any number by 2 halves it; by 3, reduces it to one third; &c. for, ſince unit is contained in any number ſo many times as that number expreſſes, therefore 2, being twice as great as unit, is contained in it half that number of times, i. e. the quotient of any number divided by 2 is half that number; and 3 being thrice as great as 1, is contained but a third part as often; or, the quotient of any number divided by 3 is a third part of that number; &c.

In multiplication we have given, the number and the multiplier; and the product is required: In diviſion we have the number and the product; and the multiplier is required: and as in multiplication the operation is to add the given number to itſelf a propoſed number of times, and reckon the amount; ſo, reverſely, in diviſion, the operation is to ſubtract the given number from the product as often as poſſible, and reckon the number of ſubtractions.

23. As this operation, performed at length, would be very tedious when the quotient is a high number, it is proper to ſhorten it by every convenient method; and for this purpoſe we may multiply the diviſor by ſome number which gives a product not greater than the dividend, and ſo ſubtract it twice, thrice, or oftner at the ſame time.

When the quotient does not conſiſt of more than one figure, the beſt way is to multiply the diviſor by the greateſt number, that does not give a product greater than the dividend; and that number is the quotient; e. g. to find how many times 7 is contained in 45, we inquire what is the greateſt multiplier of 7, that does not give a product greater than 45, which we find is 6, and 6 times 7 is 42, which, ſubtracted from 45, leaves a remainder 3; therefore 7 may be ſubtracted 6 times from 45; or, (which is the ſame thing) 45, divided by 7, gives a quotient 6, and a remainder 3.

If the diviſor do not exceed the limits of the multiplication table, we readily find the higheſt multiplier that can be uſed from that table: if it exceed thoſe limits, we may try any multiplier that we think will anſwer: if the product be found greater than the dividend, the multiplier was too great; if the remainder, after the product has been ſubtracted from the dividend, be found greater than the diviſor, the multiplier was too ſmall. In either of theſe caſes another multiplier muſt be tried.

If the diviſor be contained oftner than 9 times in the dividend, the operation requires as many ſteps as there are figures in the quotient: for inſtance, if the quotient be greater than 99, but leſs than 1000, it requires 3 ſteps. We firſt inquire how many hundred times the diviſor is contained in the dividend, and ſubtract the amount of theſe hundreds: Then we inquire how often it is contained ten times in the remainder, and ſubtract the amount of theſe tens: Laſtly, we inquire how many ſingle times it is contained in the new remainder.

It is very eaſy to find both the higheſt claſs of the quotient, and alſo what the digits of that, and the other claſſes muſt be: for, if we aſſume from the left of the dividend the leaſt part, that (conſidered independent of the reſt of the figures) conſtitutes a number not leſs than the diviſor, the claſs in which the loweſt figure of this part ſtands, (which for diſtinction ſake we will call the claſs in which the aſſumed number ſtands) is the higheſt claſs of the quotient; for, becauſe the diviſor is contained ſo many ſimple times in that aſſumed part of the dividend, (when conſidered ſimply) therefore, if that aſſumed part be tens, or hundreds, or ſome higher claſs, the diviſor is contained in it as many ten times, as many hundred times, or as many times of ſome higher denomination.

For example, ſuppoſe 5372 is to be divided by 41; 53 is the leaſt part of the dividend we can aſſume that is not leſs than 41; now 53, in this example, ſtands in the claſs of hundreds; but as often as 41 is contained in 53, ſo many hundred times is it contained in 5300, or in 5372; for it is not poſſible that the addition of this remaining part (72) of the dividend, can increaſe the firſt quotient figure, ſince this remaining part cannot be ſo great as one hundred;* therefore the higheſt digit is properly determined by conſidering how often 41 is contained in 53, and placing the

anſwer

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[*] If the aſſumed part be increaſed by 1, the diviſor (to take an extreme caſe) can be contained but once oftner, and that without any remainder; and conſequently can be contained but one ten times oftner in this increaſed number conſidered as tens; but one hundred times oftner in this increaſed number, conſidered as hundreds; &c. and that without any remainder; that is, the addition of leſs than one of the claſs in which the aſſumed number ſtands, cannot increaſe the quotient figure; but the remainder, whatever it be, is always leſs than one of the claſs in which the aſſumed number ſtands; therefore the addition of the remainder cannot affect the firſt figure of the quotient. Thus, 7 is contained in 27, 3 times; therefore

anſwer (which is 1) in the claſs of hundreds; 41 then is contained one hundred times in 5372; therefore we may ſubtract one hundred times 41 from 5372, which will leave 1272: now we are to find how many ten times 41 is contained in this remainder; in order to which, we inquire how many ſimple times 41 is contained in 127; the anſwer to which (3) muſt be placed in claſs of tens: then we ſubtract 30 times 41 from the remainder 1272, which leaves 42; and it only remains to determine how many ſimple times 41 is contained in 42; this is found to be once, a remainder being left of 1. Thus we have found how many times 41 is contained in 5372; we have found it to be 100 times, together with 30 times, together with once, or 131 times.*

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therefore in 27000, 3000 times; add one to the 27, it becomes 28; and 7 is contained in it exactly 4 times; therefore in 28000, exactly 4000 times: whatever therefore be ſubſtituted for the cyphers in 27000, 7 will not be contained more thouſand times than three thouſand times; for to increaſe the number the moſt poſſible, let nines be ſubſtituted; it becomes 27999, which is not ſo great as 28000.

* The operation is as follows:

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                 5372
                 4100 is 100 times 41
                 ----
1ſt Remainder    1272
                 1230 is 30 times 41
                 ----
2d Remainder       42
                   41 is once 41
                 ----
3d Remainder        1
                 ----   ---
                        131 The total number of ſubtractions.
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It might have happened, that the diviſor ſhould not have been contained in the firſt remainder any ten times at all; in that caſe we muſt have put a cypher in the tens place of the quotient, and have proceeded to inquire how many ſimple times the diviſor might be contained in the ſame remainder.

In ſubtracting the hundred times 41, the moſt convenient method is, only to multiply the 41 by the digit 1, and ſubtract the product from the 53, from which that digit was obtained; the next figure of the dividend (7) annexed to the remainder (12) conſtitutes the 127, by which it is to be determined the tens place of the anſwer. If this number had been leſs than 41, we muſt have placed a cypher as the next figure of the quotient, and have annexed the remaining figure of the dividend, to have determined the units place of the quotient: and, in general, it is obvious, that if we always multiply the diviſor by the ſimple digit obtained, and ſubtract the product from that part of the dividend, from which the digit was obtained, the remainder, with one figure more of the dividend annexed, conſtitutes the number, by which we muſt determine the next place of the quotient, and if it be found that a cypher muſt occupy this place, (i. e. if the number made up of the remainder and figure annexed be found leſs than the diviſor) another figure of the dividend, annexed to the ſame remainder conſtitutes the number, by which we muſt determine the ſucceeding place of the quotient, &c. Hence the following

RULE FOR DIVISION OF NUMBERS.

24. Place the dividend to the right hand of the diviſor, ſeparating them by a line: inquire how often the diviſor be contained in that number, which the higheſt one, two or more figures of the dividend expreſs, independently of the reſt; taking no more figures of the dividend than are ſufficient to make up a number not leſs than the diviſor: place the quotient figure to the right of the dividend, ſeparating them by a line. Multiply the diviſor by the quotient figure, and ſubtract the product from that part of the dividend which was employed in finding it. To the right hand of the remainder annex the next figure of the dividend, and if the remainder and this figure annexed conſtitute a number leſs than the diviſor, ſet down a cypher for the next place of the quotient, and annex another figure of the dividend; If the number made up of the remainder, and the two annexed figures be leſs than the diviſor, place another cypher in the quotient, and annex another figure of the dividend, and ſo on, till a number be formed that is not leſs than the diviſor, which (when obtained) conſider as a dividend, and inquire how often the diviſor be contained in it; ſet the anſwer in the

next place of the quotient; multiply the diviſor by this laſt obtained figure, and ſubtract the product from the laſt dividend; proceed with the remainder as before: when there are no more figures of the dividend to annex, the operation is completed.

EXAMPLES.

Diviſor Dividend Quotient
1ſt. 365) 317550 (870
2920
2555
2555
. .0

Diviſor Dividend Quotient
2d. 27) 6210038 (230001
54
81
81
0038
27
11 Remainder

Definition. The dividends by which each figure of the quotient is obtained, are called dividuals.

If it ſhould not readily appear how often the diviſor be contained in any dividual, cut off all but the firſt or higheſt figure of the diviſor, and as many figures from the dividual, and inquire how often this firſt figure is contained in the remaining part of the dividual; the anſwer is always as great as the required quotient figure, and generally too great by one or two. e. g. to find how often 365 is contained in 3175, we inquire how often 3 is contained in 31; the anſwer is 9; and ſo often is 300 contained in 3100; and nearly ſo often muſt 365 be contained in 3175.

Obſervation 1. The uſe of actually multiplying the diviſor by the quotient figure, is to find the remainders; if the diviſor be ſo ſmall, that they can be had by inſpection, this multiplication is needleſs.

EXAMPLES.

3d. 8) 83 (3 Remainder

10 Quotient

4th. 12) 1211890 (10 Remainder

100990

Obſervation 2. When cyphers occupy the right hand place or places of the diviſor, cut them off, and cut off an equal number of figures from the dividend.

For if there be one, two, &c. cyphers to the right, in the diviſor, the diviſor conſiſts of an exact number of tens, hundreds, &c. in the reſpective caſes; and therefore the quotient is not affected by the units place of the dividend; by the units and tens place; &c. reſpectively; but is the ſame as if ſo many places of the dividend were cyphers. Now if the dividend and diviſor had an equal number of cyphers in the right hand places, they clearly might be neglected in performing the operation of diviſion.

For as many times as the diviſor is contained in the dividend, ſo many times is a tenth part of the diviſor contained in a tenth part of the dividend, ſo many times is an hundredth part of the diviſor contained in an hundredth part of the dividend, &c.

The true remainder is had by annexing to the remainder ariſing from the operation, the figures of the dividend, that were cut off.

For if a tenth part of the dividend, (neglecting the value of the units) or an hundredth part, (neglecting the value of the units and tens) &c. contain a tenth part or an hundredth part, &c. of the diviſor ſo many times, with a certain remainder over, the whole dividend (neglecting the value of the ſame places) will contain the whole diviſor as many times, with ten times, one hundred times, &c. the ſame remainder, in the reſpective caſes; therefore if the dividend be conſidered in its complete value, the remainder is ten times, one hundred times, &c. the reſulting remainder, together with the part cut off; i. e. the part cut off muſt be annexed to the remainder ariſing from the operation.

Thus, if the diviſor be 500, and the dividend 3721; becauſe 37 contains 5, 7 times and 2 over, therefore 3700 contains 500, 7 times and 200 over, and 3721 contains 500, 7 times with 221 over.

EXAMPLES.

5th. 16,000) 864,023 (54. Rem. 23
80
64
64
0

6th. 7,00) 309,00 (
44 . Rem. 100

Obſervation 3. Hence to divide by 10, by 100, or, in general, by unity with any number of cyphers annexed, we need only cut off that number of figures from the right hand of the dividend; the other figures of the dividend exhibit the quotient, and the figures cut off, the remainder.

Ex. The quotient of 71023 divided by 100, is 710; the remainder 23.

The quotient of 546001 divided by 1000, is 246; the remainder 1.

The quotient of 330000 divided by 10000, is 33; the remainder 0.

25. There are no other methods than theſe four, given under the rules of addition, ſubtraction, multiplication and diviſion, by which numbers are increaſed or diminiſhed.

CHAPTER II.

DEFINITIONS.

Article 26. ALGEBRA is *Univerſal* Arithmetic; certain relations being ſuppoſed between numbers, other relations and properties are deduced by common reaſoning.

27. To abbreviate this reaſoning, arbitrary ſymbols are put to denote both the numbers reaſoned upon, and the operations ſuppoſed to take place between them: ſo that algebra may be conſidered as a kind of ſhort hand writing, which may at any time be tranſlated into words at length. Ludlam

Such a ſhort hand is of great uſe in reducing any reaſoning into writing. The argument is more readily comprehended, and the relation between diſtant ideas much more eaſily diſcerned, than when incumbered by words at length.

28. In algebra quantities are either wholly repreſented by ſymbols, or partly by ſymbols, and partly by numerical characters. So far as they are repreſented by diſſimilar ſymbols, the operations of addition, ſubtraction, multiplication and diviſion cannot actually take place: ſo far as they

are repreſented by numerical characters or ſimilar ſymbols, they may: For inſtance, if any number be repreſented by the letter *b*, and another by the letter *c*, the ſum of theſe two numbers can no otherwiſe be expreſſed, than by ſome mark denoting the addition of *c* to *b*; but, if to the ſum of *b* and 3 (expreſſed by a ſymbol) we have to add 7, we may incorporate the 7 with the 3; and it becomes the ſum of *b* and 10, expreſſed by that ſame ſymbol: Again, if 2*b* be defined to ſignify twice the number repreſented by *b*, and it be required to multiply 2*b* by the number *c*, ſuch multiplication muſt be expreſſed by ſome ſign; but if it be required to multiply 2*b* by 3, we need only change the 2 to 6, and write 6*b*, which may denote 6 times *b*, (or 3 times twice *b*) if 2*b* denotes twice *b*.

DEFINITIONS AND EXPLANATIONS OF SIGNS.

29. Numbers are repreſented by the letters of the alphabet.

30. The ſign + (*plus*) denotes that the quantity, to which it is prefixed, is to be added.

Thus, $a+b$ ſignifies, that the quantity repreſented by *b* is to be added to that, repreſented by *a*, and is read, *a plus b*.

If *a* repreſent 5, and *b*, 7, $a+b$, repreſents 12.

31. Quantities

31. Quantities to be added are called poſitive quantities.

32. The ſign $-$ *(minus)* denotes that the quantity to which it is prefixed, is to be ſubtracted.

Thus $a-b$ ſignifies that b is to be ſubtracted from a, and is read, *a minus b.*

If a repreſent 7, and b, 5, $a-b$ repreſents 2.

33. Quantities to be ſubtracted are called negative quantities.

Note. The characters $+$ and $-$, from their extenſive uſe in algebra, are called "The Signs."

33. The ſign × *(into)* denotes that the quantities, between which it ſtands, are to be multiplied * together.

Thus $a\times b$ ſignifies that a is to be multiplied by b.

If a be 2, and b, 3, $a\times b$ repreſents 6.

35. A full point is frequently uſed for the ſame purpoſe.

Thus, $1\times2\times3$, and 1. 2. 3 ſignify the ſame thing.

36. The multiplication of ſimple factors is uſually denoted by their juxta poſition, without any other ſign.

Thus, ab ſignifies the ſame as $a\times b$.

37. In products where the ſame factor is repeated, it is uſual to write down the repeated factor

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* We ſay multiplied together, becauſe the product a multiplied by b, is equal to the product b multiplied by a. (See Art. 70.)

only once, and place a number at the head, indicative of the number of like factors.

Thus, xx is written x^2; xxx is written x^3; &c. $2aaxxxx$ is written $2a^2x^4$; &c. &c.

The ſign of multiplication is read (where neceſſary) by the word *into:* Thus 2×3, is read *two into three.*

38. The ſign $\div$ (*divided by*, or *by*) denotes that the former of the quantities, between which it is placed, is to be divided by the latter.

Thus, $a \div b$, ſignifies that a is to be divided by b, and is read, *a divided by b*, or, *a by b.*

If a be 6, and b, 3, $a \div b$ repreſents 2.

39. The diviſion of one quantity by another, is alſo repreſented by placing the diviſor under the dividend, with a line between them.

Thus $\frac{a}{b}$ ſignifies a divided by b.

40. The ſign $\sim$ between two quantities denotes their difference.

Thus, $a \sim x$, ſignifies the exceſs of x above a, or the exceſs of a above x, according as x or a be the greater.

41. A line drawn over ſeveral quantities denotes that they are to be taken collectively, and is called a vinculum.

Thus $\overline{a+b} \times \overline{c+d}$, or $\overline{a+b} \cdot \overline{c+d}$, ſignifies that the ſum of a and b is to be multiplied by the ſum of c and d; $\overline{a+b} \div \overline{c+d}$, ſignifies that the ſum of a and b is to be divided by the ſum of c and d.

If diviſion be repreſented according to the ſecond method, there is no occaſion for the vinculum.

Thus $\frac{a+b}{c+d}$ is the ſame as $\overline{a \times b} \div \overline{c \times d}$.

Again, $\overline{a-b}+\overline{c-d}$, ſignifies that the quantity $c-d$, is to be added to the quantity $a-b$; $\overline{a+b}-\overline{c-d}$, ſignifies that the quantity $c-d$ is to be ſubtracted from the ſum of a and b.

42. Inſtead of the vinculum, a parentheſis is ſometimes uſed.

Thus, $\overline{a+b} \times \overline{c+d}$, and $(a+b) \times (c \times d)$ ſignify the ſame thing.

43. The ſign $=$ *(equal to)* denotes that the quantities, between which it is placed, are equal to each other.

Thus $a+b=c$, ſignifies that the quantity $a+b$ is equal to the quantity c; $ax-by=cd+d^2$, ſignifies that the quantity $ax-by=$, is equal to the quantity $cd+d^2$.

44. The number prefixed to any quantity, which ſhews how often that quantity is to be taken, is called the coefficient of the term in which it is found.

Thus, 7, 4 and 3 are called the coefficients of $7ax$, $4by$ and $3z$ reſpectively.

Unity may be ſaid to be the coefficient of a term or expreſſion where no number is prefixed to the letters; becauſe ſuch term or expreſſion repreſents a quantity ſimply taken once.

Thus, 1 is the coefficient of a, or $\overline{a+b}$, becauſe a is $1 \times a$, and $a+b$ is $1 \times \overline{a+b}$. Theſe numbers are ſometimes repreſented by letters, which are then called coefficients.

45. Similar, or like algebraical quantities, are ſuch as differ only in their numerical part.

Thus $4a$, $5ab$, $11a^2$, $a^2 b^2 c$, are reſpectively ſimilar to $6a$, $9ab$, $9a^2$ $17a^2b^2c$.

46. Unlike quantities are different combinations of letters.

Thus, a, a^2, $2ab$, $a^2 b$, ab^2, abc, &c. are all unlike.

47. A quantity is ſaid to be a multiple of another, when it contains that other a certain number of times exactly.

Thus, $15a$ is a multiple of $3a$, containing $3a$ exactly 5 times.

48. A quantity is ſaid to be a meaſure of another, when it is contained in that other a certain number of times exactly.

Thus, $3a$ is a meaſure of $15a$, being contained in $15a$ exactly 5 times.

49. A ſimple algebraic quantity is one, which conſiſts of a ſingle term; as a, or ab, or $\frac{ac}{b}$, &c.

50. A compound quantity is one that conſiſts of ſeveral terms; as $a+b$; $2a-3c+4d$; &c.

51. A compound quantity conſiſting of two terms, is called a binomial; of three terms, a trinomial, &c. of many terms, a multinomial.

52. A binomial, one of whoſe terms is poſitive, and the other negative, is called a reſidual.

53. The ſign $\therefore$ is uſed as an abbreviation for the word *therefore.*

54. Aſteriſks are frequently put to denote vacant places.

Thus, if in the ſeries of natural numbers 1, 2, 3, 4, 5, 6, the 3 be omitted in any particular caſe, the ſeries may be expreſſed thus, 1, 2, * 4, 5, 6.

CHAPTER III.

OF ADDITION AND SUBTRACTION.

ARTICLE 55. SINCE the addition of any quantity is expreſſed by ſetting it down with the ſign + prefixed, the ſum of a and b is written $a+b$; the ſum of a and $\overline{b-c}$ (where b is ſuppoſed greater than c) is written $a+\overline{b-c}$, or $a+b-c$; for whether we firſt ſubtract c from b, and add the remainder to a, or add to a the whole of b, and from their ſum ſubtract c, the reſult will evidently be the ſame.*

In the ſame manner, if to $a-b$, $c-d$ is to be added, (where again b is ſuppoſed leſs than a, and d leſs than c) the ſum is $\overline{a-b}+\overline{c-d}$, or $\overline{a-b}+c-d$, or $a-b+c-d$. and, in general,

56. The ſum of any compound, or ſimple quantities, may be expreſſed by ſetting down each term with its ſign prefixed; and the reſulting expreſſion may be conſidered either as the ſum of all the poſitive terms diminiſhed by the ſum of all the negative terms, or as the ſums, or differences, or ſums and

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* b exceeds $b-c$ by c ∴ $a+b$ exceeds $a+\overline{b-c}$ by c ∴ $a+b-c=a+\overline{b-c}$.

differences of any combinations of terms whatſoever, taken in any order.

Thus the above expreſſion $a-b+c-d$ is either the ſum of a and c diminiſhed by the ſum of b and d; or it is the difference of a and b encreaſed by the difference of c and d; or it is the difference of a and d added to or ſubtracted from c, (according as a be greater or leſs than d) and the reſult diminiſhed by b; &c.

Hence it appears, that the order in which the terms are ſet down does not affect the value of the expreſſion, provided each term have its proper ſign prefixed.

57. Alſo, ſince the differences between quantities are denoted by the ſign $-$, if of the two quantities a and b, a be the greater, their difference is expreſſed thus, $a-b$; if the quantities be a and $\overline{b-c}$, (c being ſuppoſed leſs than b, and the difference between b and c leſs than a) their difference is written $a-\overline{b-c}$, or $a-b+c$, which is the ſame; for the difference of any two quantities is the ſame as the difference of thoſe two quantities increaſed by any common quantity $\therefore$ a diminiſhed by $b-c$, $(a-\overline{b-c})$, is the ſame as $a+c$ diminiſhed by b, $(a-b+c)$, ſince $a+c$ exceeds a, and b, $(=b-c+c)$ exceeds $b-c$, by the common quantity c.

Here $b-c$ may repreſent any compound quantity whatſoever, conſiſting of poſitive and negative terms, ſince any ſuch quantity is equal to the ſum of all its poſitive terms, diminiſhed by the ſum of all its negative terms.

58. From this reaſoning it appears, that if by addition were meant the increaſing of a quantity by ſome other quantity, the rule for the addition of quantities would be, Set down all the quantities, prefixing to each term its proper ſign.

59. And if by ſubtraction were meant the diminiſhing a quantity by ſome other (which other cannot be greater than the quantity to be diminiſhed, ſince from any quantity there can be taken no more than the whole) the rule for ſubtracting quantities would be, Connect the ſubtrahend with the quantity from which it is to be ſubtracted, prefixing $-$ to each of its poſitive terms, and $+$ to each of its negative terms; or, (which is the ſame thing) Change the ſign of each term in the ſubtrahend, and connect it as in addition.

60. But the word *addition* is uſed by writers on algebra in a more extended ſenſe. Quantities are imagined to be of two kinds, poſitive and negative, poſitive diſtinguiſhed by the ſign $+$, negative by the ſign $-$; having an oppoſite effect in all expreſſions, into which they enter as parts: thus, as $2a+a$ (or $3a$) is the ſum of $+2a$ and $+a$, or of $2a$ and a, ſo $2a-a$, (or a) is conceived to be the ſum of $2a$ and $-a$; $a-b$ is the ſum of a and $-b$; $a+b-c$ is the ſum of a and $b-c$, whether b be greater than c or not.

61. If b be leſs than a, $a-b$, or the ſum of a and $-b$ is *really* the difference of a and b, but if b be greater than a, ſuppoſe equal to $a+n$, then $a-b$, or $a-a-n$ is equal to $-n$; or $a-b$ then ſignifies the negative difference between a and b.

Thus, the real difference between 3 and 5 being 2, $3-5$, or $3-3-2$, ſignifies -2.

62. If to $-a-b$, (the ſum of $-a$ and $-b$), $+b$ be added, the ſum is $-a-b+b$, or $-a$, which is a leſs negative quantity than $-a-b$, but may be conceived to be greater than $-a-b$, if by greater be meant leſs removed from poſitive quantity; and under this idea poſitive quantities may be defined ſuch as increaſe, and negative quantities ſuch as decreaſe the value of any expreſſion with which they are connected.

63. In an analogous manner is the meaning of the word *ſubtraction* extended by writers on algebra; and as the buſineſs of ſubtraction, as explained in chapter 1. is to take away from a given quantity ſome other quantity which forms a part of it, ſo alſo is it of algebraical ſubtraction.

Thus, to ſubtract b from $a+b$, means to take away the part b; to ſubtract $-b$ from $a-b$, means to take away the part $-b$; the part left in both theſe caſes is the ſame, viz. a; and the ſame would have been the reſult, had the quantity b, with its ſign changed, been added in each caſe, i. e. $-b$ in the firſt, and $+b$ in the ſecond caſe.

64. Any quantity b or $-b$ may be conſidered as a part of any other quantity a, though it be not found among its terms; for $a=a+b-b$, therefore if $-b$ be ſubtracted from a, or from $a+b-b$, according to the above idea of ſubtraction, the reſult will be $a+b$; if $\overline{b-c}$ be ſubtracted from a, or from $a+b-c-b+c$, the reſult will be $a-b+c$; and in general

The ſubtraction of any quantity whatſoever will give the ſame reſult as the addition of the ſame terms with changed ſigns.

65. Let the difference between b and c be n; then $a+b-c$, if b be greater than c, is the ſum of a and n; it is alſo the remainder which is left after ſubtracting $-n$ from a; but if b be leſs than c, $b-c$ is $-n$; and $a+b-c$ is then the ſum of a and $-n$, or the remainder which is left after ſubtracting n from a.

Again, if b be greater than c, $a-\overline{b-c}$ is the remainder left after ſubtracting n from a; it is alſo the ſum of a and $-n$; but if b be leſs than c, $b-c$ is $-n$, and $a-\overline{b-c}$ is the remainder left after ſubtracting $-n$ from a; or the ſum of a and n.

66. From what has been ſaid above, it is evident that algebraic addition may be defined

"Connecting quantities together, prefixing to each term its proper ſign."

And that algebraic ſubtraction may be defined

"The finding that quantity, which added to the ſubtrahend, forms the quantity from which the ſubtraction is to be made."

67. In addition and ſubtraction the reſulting expreſſions may be ſhortened, when any of the terms conſiſt of the ſame literal part, by actually incorporating ſuch terms together.

Thus, $a+a$, is more conveniently expreſſed by $2a$; $a-2a$, by $-a$; $-a-\overline{b-4a}$ (or $-a-b+4a$) by $3a-b$; &c. &c.

68. To facilitate the actual performance of addition, and ſubtraction, the general rules for the different caſes, are here ſubjoined; the reaſon of each rule will readily appear from what has been ſaid above.

ADDITION.

ADDITION.

Case 1. To add quantities which are alike, and have like ſigns.

Rule. Add together the coefficients; to their ſum prefix the common ſign; and ſubjoin the common letter or letters.

Examples.

$$\begin{array}{ll} \text{To} & +5a \\ \text{Add} & +4a \\ \hline \text{Sum} & 9a \\ \hline \end{array} \qquad \begin{array}{ll} \text{To} & -6b \\ \text{Add} & -2b \\ \hline \text{Sum} & -8b \\ \hline \end{array} \qquad \begin{array}{ll} \text{To} & a+b \\ \text{Add} & 3a+5b \\ \hline \text{Sum} & 4a+6b \\ \hline \end{array}$$

$$\begin{array}{ll} \text{To} & 3a-4b \\ \text{Add} & 5a-8b \\ \hline \text{Sum} & 8a-12b \\ \hline \end{array} \qquad \begin{array}{ll} \text{To} & a-b+x \\ \text{Add} & a-2b \\ \hline \text{Sum} & 2a-3b+x \\ \hline \end{array}$$

Case 2. To add quantities that are alike, but have unlike ſigns.

Rule. Subtract the leſs coefficients from the greater; prefix the ſign of the greater to the remainder, and ſubjoin the common letter or letters.

Examples.

EXAMPLES.

$$\begin{array}{lr}\text{To} & -4a\\ \text{Add} & +7a\\ \cline{2-2} \text{Sum} & +3a\\ \cline{2-2}\end{array}\qquad\qquad\begin{array}{lr}\text{To} & +5b-6c\\ \text{Add} & +3b+8c\\ \cline{2-2} \text{Sum} & 8b+2c\\ \cline{2-2}\end{array}$$

$$\begin{array}{lr}\text{To} & a+6x-5y-8\\ \text{Add} & -5a-4x+5y+3\\ \cline{2-2} \text{Sum} & -4a+2x \;\ast\; -5\\ \cline{2-2}\end{array}\qquad\qquad\begin{array}{lr}\text{To} & 2a-3b\\ \text{Add} & -2a+3b\\ \cline{2-2} & \ast\quad\ast\\ \cline{2-2}\end{array}$$

If there be more than two quantities to be added together, firſt collect the poſitive into one ſum, and the negative into another (by caſe 1.): then add the two reſulting ſums together, by the preſent caſe.

$$\begin{array}{lr}\text{Ex.} & +8a\\ & -7a\\ & +10a\\ & -12a\\ \cline{2-2} \text{Sum of the poſitive} & +18a\\ \text{Sum of the negative} & -19a\\ \cline{2-2} \text{Sum of all} & -\quad a\\ \cline{2-2}\end{array}$$

Case 3. To add quantities that are unlike.

Rule. Set down all the terms one after another, with their ſigns prefixed.

Examples.

$$\begin{array}{ll} \text{To} & +2a \\ \text{Add} & +3b \\ \hline & 2a+3b \\ \hline \end{array} \qquad \begin{array}{ll} \text{To} & +3a \\ \text{Add} & -4x \\ \hline & +3a-4x \\ \hline \end{array} \qquad \begin{array}{ll} \text{To} & 4a+4b+3c \\ \text{Add} & -x-4y+3z \\ \hline \multicolumn{2}{c}{4a+4b+3c-x-4y+3z} \\ \hline \end{array}$$

Example to all the Cases.

$$\begin{array}{ll} & 2a+3b+c \\ & 5a-7b-2d \\ & 3a+4b+3e \\ & 4a-7c \\ \hline \text{Sum} & 14a-6c-2d+3e \\ \hline \end{array}$$

SUBTRACTION.

General Rule. Change the ſigns of the quantity to be ſubtracted to the contrary ſigns; add it ſo changed to the quantity from which it was to be ſubtracted (by the preceding rules): the ſum reſulting from this addition is the remainder required.

EXAMPLES.

$$\begin{array}{lr} \text{From} & 5a \\ \text{Sub.} & 3a \\ \hline & 2a \\ \hline \end{array} \qquad \begin{array}{lr} \text{From} & 8a-7b \\ \text{Sub.} & -3a+4b \\ \hline & 11a-11b \\ \hline \end{array} \qquad \begin{array}{lr} \text{From} & 3x \\ \text{Sub.} & -2a \\ \hline & 3x+2a \\ \hline \end{array}$$

$$\begin{array}{ll} \text{From} & a-3x+5y \\ \text{Sub.} & 6a+4x-2d \\ \hline \multicolumn{2}{l}{-5a-7x+5y+2d} \\ \hline \end{array} \qquad \begin{array}{ll} \text{From} & a-2b+d-3e \\ \text{Sub.} & a-2b+x \\ \hline & *\ \ *+d-x-3e \\ \hline \end{array}$$

CHAPTER IV.

OF MULTIPLICATION.

ARTICLE 69. IF the multiplier and multiplicand be divided into parts, and each part of the multiplicand be multiplied by each part of the multiplier, the ſum of all the reſulting products is the product of the whole multiplicand multiplied by the whole multiplier.

Thus, if $a+b+c$ be the multiplicand, and $d+e$ be the multiplier, the product $\overline{a+b+c}\times\overline{d+e}$ is $ad+bd+cd+ae+be+ce$.

70. Hence it follows that in all caſes, if the multiplicand be made the multiplier, and the multiplier the multiplicand, the product will be the ſame as before; for let m denote any multiplier whatſoever, and n any multiplicand; then as n is $1+1+1+$ &c. to n terms, n multiplied by m, or n taken m times, is $1\times m+1\times m+1\times m+$ &c. to n terms, or $m+m+m+$ &c. to n terms (by laſt article); but $m+m+m+$ &c. to n terms is m taken n times, or m multipled by n,

$\therefore$ n taken m times is equal to m taken n times.

71. It follows alſo that any product mn, multiplied by r, is equal to the product mr multiplied by n, or to the product nr multiplied by m.

For mn is $m+m+$ &c. to n terms;

$\therefore mn$ multiplied by r is $mr+mr+$ &c. to n terms;

But $mr+mr+mr+$ &c. to n terms is mr taken n times;

$\therefore mn$ taken r times $= mr$ taken n times;

And in the ſame manner, becauſe mn is $n+n+$ &c. to m terms, it follows that $mn \times r = nr \times m$.

72. It follows alſo that $mnr \times p = mnp \times r = nrp \times m =$ &c. $= mn \times rp = mp \times nr =$ &c.

For mnr is $mn+mn+$ &c. to r terms;

$\therefore mnr \times p = mnp + mnp +$ &c. to r terms;

But $mnp + mnp +$ &c. to r terms $= mnp \times r$;

$\therefore mnr \times p = mnp \times r$:

Again, mnr (by laſt article) $= m+m+$ &c. to nr terms; $\therefore mnr \times p = mp + mp +$ &c. to nr terms; but $mp + mp +$ &c. to nr terms $= mp \times nr$, $\therefore mnr \times p = mp \times nr$; &c. &c.

73. By continuing the ſame method of reaſoning, it appears that, in general, whatever be the number of factors, the product may be conſidered as the product of any two combinations of them; and becauſe the products, which theſe combinations repreſent, may again be conſidered as reſulting from any combinations of their factors, it follows that

A product $a.b.c.d.e.f.$ &c. may be conſidered as the product ab multiplied by cd, and the reſult multiplied by ef &c. or as a multiplied by the product cf, and the reſult multiplied by $b\,d\,e$ &c. or any other way in which the factors can be combined; in other words, a product depends only on the factors, not on the order on which the multiplication is carried on.

Note. The numeral part of a product is uſually prefixed to the other factors.

Thus, $2a \times 3b$ is not written $2a3b$, but $2.3ab$, or $6ab$.

74. The general deſignation of a product being by placing the ſign ×, or a full point between the factors, or, if the factors be ſimple quantities, by juxta poſition; it appears (Art. 69.) that,

When the factors are compound quantities, all whoſe terms are poſitive, the product is alſo truly repreſented by the ſum of the products of every two of the parts.

75. When one or both of the factors are compound quantities, conſiſting of the exceſs of the ſum of certain poſitive terms above the ſum of certain negative terms, the product is truly repreſented by the collected products of each term of the multiplier, multiplied by each term of the multiplicand, the ſign + being prefixed to the product of any two terms, that have the ſame ſign, and the ſign − to the product of any two terms that have different ſigns. For,

1ſt. Let the factors be $a-b$ and c; the quantity $\overline{a-b}$, taken once, or $a-b$, is leſs than a by the quantity b; and $\overline{a-b}$ taken twice is leſs than a taken twice by the quantity $2b$; for $\overline{a-b}+\overline{a-b}=2a-2b$: in the ſame manner it appears that $\overline{a-b}$, taken c times, is leſs than a taken c times by the quantity bc; but a taken c times is ac, $\therefore a-b$ taken c times is $ac-bc$, conformably to the above law.

2d. Let the factors be $a-b$ and $c+d$; by Art. 69. $\overline{a-b}\times\overline{c+d}=\overline{a-b}\times c+\overline{a-b}\times d$; but (by caſe 1ſt.) $\overline{a-b}\times c=ac-bc$; and $\overline{a-b}\times d=ad-bd$ $\therefore \overline{a-b}\times\overline{c+d}=ac-bc+\overline{ad-bd}=ac-bc+ad-bd$, conformably to the above law; and it is evident that the ſame conformity would be found, if inſtead of the factor $c+d$, we employed a factor conſiſting of more terms.

3d. Let the factors be $a-b$ and $c-d$; (by caſe 1ſt.) $a-b$ taken c times is $ac-bc$; but $c-d$ is leſs than c by the quantity d, $\therefore$ $\overline{c-d}$ times $\overline{a-b}$ is leſs than c times $\overline{a-b}$ by the quantity d times $\overline{a-b}$; i. e. by the quantity $ad-bd$; $\therefore \overline{a-b}\times\overline{c-d}=ac-bc-\overline{ad-bd}=ac-bc-ad+bd$, conformably to the ſame law.

And ſince any compound quantity whatſoever, wherein poſitive and negative terms are found, may be divided into two parts, the one conſiſting of all the poſitive terms, and the other conſiſting of all the negative terms, theſe three caſes eſtabliſh the law univerſally.

76. Hitherto the reaſoning has been confined to real quantities, i. e. either to ſimple poſitive quantities, or to compound quantities, conſiſting of poſitive terms only, or to compound quantities, the ſum of whoſe poſitive terms exceed the ſum of their negative terms: but it is evident that the definition of multiplication ſerves equally to determine the product of any imaginary quantity, when multiplied by a real quantity.

Thus if $-a$ be conceived as a quantity, the double of $-a$, or $-a$ multiplied by 2, will be deſignated by a negative quantity twice as great, i. e. by $-2a$; the triple of $-a$, or $-a \times 3$ will be deſignated by $-3a$; and, in general, n times $-a$ will be deſignated by $-na$.

77. The ſame law which obtains when both factors are poſitive, ſerves alſo for determining the product ariſing from any negative multiplicand.

For let the multiplicand, whether it be ſimple or compound, be equal to $-A$; and let the multiplier be equal to B; then, becauſe from the application of this law to any multiplicand the ſame product reſults as from the application of it to that multiplicand with the ſign of each term changed, except that in the latter product, the ſign of each term will be found changed, it follows that, ſince, if we change the ſigns of this negative multiplicand, thereby making it equal to $+A$,

$+A$, and then multiply it by B, according to that law, the reſult will be equal to AB, therefore, if we multiply this negative multiplicand by B, according to that law, the reſult will be equal to $-AB$.

Thus, if $a-b=-n$, and conſequently $b-a=n$, ſince $\overline{b-a}\times c$ $(=nc)$ is $bc-ac$, it follows that $\overline{a-b}\times c$, (which is $-nc$) is $ac-bc$, determined by the ſame law.

78. Article 76 determines what the product of a negative multiplicand muſt be, when multiplied by any real quantity; and in art. 77, it is ſhewn that ſuch product is always to be obtained by the application of the law in art. 75; but neither from them, nor from the preceding articles, can any thing be deduced, proving what the product of any quantity, real or imaginary may be, when multiplied by a negative quantity, ſince whatſoever number of terms a quantity be taken, that number cannot poſſibly be other than a real poſitive number; or in other words, the definition of multiplication does not include negative multipliers.

Neverthelеſs, from the conſideration of the nature and analogies of multiplication, if operations with abſtract negative quantities be allowed, *this* may be determined—viz. what definition of multiplication by negative quantities, will be moſt convenient for algebraical operations.

79. Suppoſe that in the ſolution of ſome problem, it ſhould ſeem expedient to multiply ſome quantity c by a compound quantity $x-y$, where it is not known whether x or y be the greater; if x be the greater, the product is $cx-cy$, determined by the above law. Unleſs therefore $cx-cy$ be allowed to be the product in either caſe, the operation would be at a ſtand; let us conſider, then, what will be the reſult of extending this law to all caſes of negative multipliers, and whether the products, ſo determined, will, when the multipliers are compound, agree with the products which ariſe from the application of the ſame law, when ſimple negative quantities, of equal value, are ſubſtituted in the room of theſe compound multipliers.

Firſt, let the multiplicand, whether ſimple or compound, be a poſitive quantity and equal to A; and let the multiplier, whether ſimple or compound, be equal to $-B$: Then, ſince it can make no difference in the reſult, whether A be conſidered as the multiplicand, and $-B$ the multiplier, or $-B$ be conſidered as the multiplicand, and A the multiplier, provided that the products be determined in both caſes by the above law, and that $-B$ multiplied by A, according to that law, produces a quantity equal to $-AB$, it follows that A multiplied by $-B$, according to that law, produces a quantity equal to $-AB$.

Thus, if $a-b=-n$, c multiplied by $a-b$, according to that law, produces $ac-bc=ac-ac-nc=-nc$, the product of c multiplied by $-n$, according to the ſame law.

Secondly, let the multiplicand be a negative quantity, and equal to $-A$; and let the multiplier be equal to $-B$; then, ſince if we change the ſign of each term of the multiplier, thereby making it equal to $+B$, and then multiply according to the above law, the product will be equal to $-AB$, and that if we multiply by any quantity, and afterwards by that quantity with all its ſigns changed, ſtill obſerving the ſame law, the two products will conſiſt of the ſame terms, but with changed ſigns, it follows, that if, according to the above law, we multiply a negative multiplied equal to $-A$, by a negative multiplier equal to $-B$, the reſult will be equal to $+AB$.

Thus, if $a-b=-n$, ſince c multiplied by $a-b$ is $ac-bc=-nc$, $-c$ multiplied by $a-b$, by the ſame law, is $bc-ac=+nc$, the product of $-c$ multiplied by $-n$, according to the ſame law.

80. It appears then, that if we determine the product of any multiplicand whatſoever, when multiplied by a negative multiplier, by the above law, the products, when the multiplicand and multiplier, one or both, be compound, will be the ſame as would reſult from the application of the ſame law to ſimple quantities of equal value, ſubſtituted in the room of theſe compound quantities.

81. Moreover it appears that ſuch multiplication by negative quantities may be defined

"Subtracting the multiplicand, * as often as there are units in the multiplier."

For $-AB$, the product of A into $-B$, is A ſubtracted as often as there are units in B; and $+AB$, the product of $-A$ into $-B$, is $-A$ ſubtracted as often as there units in B.

82. It is to be obſerved, that $\pm A$ multiplied into $-B$, the product being determined as above, is equal to $-B$ multiplied into $\pm A$, analogous to the property of real factors.

83. There are other analogies to be derived from algebraical operations, ſhewing that, if multiplication by abſtract negative quantities be allowed, the above definition muſt neceſſarily be admitted; but as they cannot be anticipated, we may conclude this diſcuſſion with obſerving that, even if the above reaſonings were nugatory, yet if ſuch a definition of multiplication by negative quantities be aſſumed, (and ſuch an aſſumption is by all writers on algebra, who admit abſtract negative quantities, either made or implied) whatever deductions be fairly drawn from it, can cauſe no error to ariſe in algebraical computations.

84. The reſult of the above remarks on multiplication, ſo far as concerns practiſe, is contained in the following

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* Conſidered abſtracted from its ſign.

Univerſal Rule for Multiplication.

Multiply every term of the multiplicand by each term of the multiplier, one after another, prefixing + to the product of any two terms, that have like ſigns, and – to the product of any two terms that have unlike ſigns; collect all the products into one ſum; that ſum is the product required.

Note. It is convenient to arrange the terms in ſuch a manner, that like terms in the product may ſtand one under the other; in order that ſuch terms may readily be collected together for the final expreſſion.

EXAMPLES.

1ſt.

$$\begin{array}{ll} \text{Mul}^{d} & a+x \\ \text{Mul}^{r} & a+x \\ \hline & a^2+ax \\ & \qquad +ax+x^2 \\ \hline \text{Prod.} & a^2+2ax+x^2 \\ \hline \end{array}$$

2d.

$$\begin{array}{ll} \text{Mul}^{d} & -a-x \\ \text{Mul}^{r} & -a-x \\ \hline & a^2+ax \\ & \qquad +ax+x^2 \\ \hline \text{Prod.} & a^2+2ax+x^2 \\ \hline \end{array}$$

3d

$$\begin{array}{l} a+x \\ a-x \\ \hline a^2+ax \\ \qquad -ax-x^2 \\ \hline a^2 \quad \ast \quad -x^2 \\ \hline \end{array}$$

4th.

$$\begin{array}{l} -x+a \\ x+a \\ \hline -x^2+ax \\ \qquad -ax+a^2 \\ \hline -x^2 \quad \ast \quad +a^2 \\ \hline \end{array}$$

5th.
$$\begin{array}{l} a^2+ax+x^2 \\ a-x \\ \hline a^3+a^2x+ax^2 \\ \quad\quad -a^2x-ax^2-x^3 \\ \hline a^3 \quad * \quad * \quad -x^3 \\ \hline \end{array}$$

6th.
$$\begin{array}{l} a^2-ax+x^2 \\ a+x \\ \hline a^3-a^2x+ax^2 \\ \quad\quad +a^2x-ax^2+x^3 \\ \hline a^3 \quad * \quad * \quad +x^3 \\ \hline \end{array}$$

7th.
$$\begin{array}{l} x^2-4x+8 \\ x^2+4x+8 \\ \hline x^4-4x^3+8x^2 \\ \quad\quad +4x^3-16x^2+32x \\ \quad\quad\quad\quad +8x^2-32x+64 \\ \hline x^4 \quad * \quad * \quad * \quad +64 \\ \hline \end{array}$$

8th.
$$\begin{array}{l} 2a+3b \\ 3ax-4by \\ \hline 6a^2x+9abx-8aby-12b^2y \\ \hline \end{array}$$

9th.
$$\begin{array}{l} x^2-ax-bx+ab \\ x-c \\ \hline x^3-ax^2-bx^2+abx \\ \quad\quad\quad -cx^2+acx+bcx-abc \\ \hline x^3-ax^2-bx^2-cx^2+abx+acx+bcx-abc \\ \hline \end{array}$$

85. Any compound expreſſion, each of whoſe terms is the product of a common quantity, multiplied into other factors, is equal to that common quantity, multiplied into the algebraical ſum of all the other factors.

Thus, $ax+bx$, or $x\times a + x\times b, = x\times\overline{a+b}$;
$ax-bx$, or $x\times a + x\times -b, = x\times\overline{a-b}$;
$a^3by-a^2by^2+aby^3-b^4y = by\times\overline{a^3-a^2b+ab^2-b^3}$;
$ax-x = x\times\overline{a-1}$.

The collecting of terms together in this manner is often uſeful in point of perſpicuity, eſpecially where the other factors are to be conſidered as co-efficients of the common quantity.

Ex. 9th would, according to this method, be written as follows:

$$\begin{array}{l}
x^2-\overline{a+b}.x+ab \\
x-c \\
\hline
x^3-\overline{a+b}.x^2+abx \\
\qquad -c\,x^2+\overline{ac+bc}.x-abc \\
\hline
x^3-\overline{a+b+c}.x^2+\overline{ab+ac+bc}.x-abc \\
\hline
\end{array}$$

CHAPTER V. PART I.

OF DIVISION.

ARTICLE 86. If the dividend be the product of the diviſor multiplied into any quantity, the quotient, which in all caſes may be repreſented by the notation in Art. 38 and 39, is that quantity.

Thus, $\frac{ab}{a}$ is b; $\frac{a}{a}$ is 1; $\frac{4ax}{2a}$ is $2x$; $\frac{ab+ac}{a}$, or $\frac{\overline{b+c}.a}{a}$ is $b+c$; $\frac{ab-ac}{a}$ or $\frac{\overline{b-c}.a}{a}$ is $b-c$; $\frac{ab+ac}{b+c}$ is a; $\frac{ab-ac}{b-c}$ is a; $\frac{a^2-2ab+b^2}{a-b}$ or $\frac{\overline{a-b}\times\overline{a-b}}{a-b}$ is $a-b$.

87. Theſe examples may be divided into three caſes.

1ſt. Where both diviſor and dividend are ſimple.

2d. Where the dividend is compound.

3d. Where both diviſor and dividend are compound.

88. In the firſt caſe the qnotient is eaſily obtained by inſpection; for we have only to expunge the diviſor from among the factors of the

dividend; the remaining factors form the quotient.

If the numeral parts of the diviſor and dividend be large, the numeral part of the quotient may not perhaps readily appear; the factors being incorporated together; but the literal part of the quotient being determined, the numeral part may be had by actually dividing the coefficient of the dividend by the coefficient of the diviſor.

Thus, $\frac{84ay}{6y}$ does not immediately appear to be $\frac{6 \times 14ay,}{6y}$ but is found to be ſo upon dividing 84 by 6.

89. In the ſecond caſe, each term of the dividend muſt be conſidered ſeparately, and the part of the quotient determined from each, muſt be that quantity, which multiplied into the diviſor produces that term.

It is evident, that in this caſe the quotient will conſiſt of as many terms as the dividend; and that from each negative term in the dividend will be obtained a negative term in the quotient.

90. In order to explain the method of finding the quotient in the third caſe, in which both diviſor and dividend are compound quantities, let it be required to divide $a^2 - 2ab + b^2$ by $a - b$: placing the quantities as in diviſion of numbers,

$a - b$

$$\begin{array}{rl} a-b\ ) & a^2-2ab+b^2\ (\ a-b \\ & a^2-ab \\ \hline & \quad -ab+b^2 \\ & \quad -ab+b^2 \\ \hline & \quad\quad \cdot\ \cdot \\ \hline \end{array}$$

Upon inſpection it appears, that a muſt be part of the quotient, ſince no other quantity multiplied into either of the terms of the diviſor can produce a^2; upon multiplying the diviſor into a, the product is found to be a^2-ab, which, ſubtracted from the dividend, leaves $-ab+b^2$; to a therefore, muſt be added ſome quantity, which, multiplied into $a-b$, produces either the whole, or part of this remainder: now there is no quantity but $-b$, which, when multiplied into either of the terms of the diviſor, can produce $-ab$; therefore $-b$ is another part of the quotient; upon multiplying the diviſor by $-b$, the product is found to be $-ab+b^2$, the whole remainder: ſince then $a-b$, multiplied into the diviſor, has produced the whole of the dividend, $a-b$ in the quotient.

91. If a quantity of any kind be a multiple of another quantity of the ſame kind, the former may be divided by the latter; hence diviſion may take place between two negative quantities. And the quotient will be the ſame as the quotient of thoſe two quantities taken poſitively.

Thus, $-a$ is contained in $-4a$ four times, or, $-4a \div -a = 4$; $\frac{-ab}{-b} = a$; $\frac{-ab-ac}{-a} = b+c$; $\frac{ab-ac}{b-c} = a$, where, if the dividend $(a \times \overline{b-c})$ be negative, it is becauſe $\overline{b-c}$, the diviſor, is negative. And the quotients of theſe negative quantities are evidently found in the ſame manner as the quotients of poſitive quantities, viz. by inquiring what expreſſion multiplied into the diviſor, according to the general rule for multiplication, will produce the dividend.

92. Between two quantities, one of which is poſitive, and the other negative, diviſion of the above kind cannot take place; for of two ſuch quantities, one cannot be ſaid to be greater, equal to, or leſs than the other, they not being of the ſame kind; thus, $2a$ is not contained any number of times in $-4a$, nor $-2a$ in $4a$; nevertheleſs ſuch quantities are ſubject to algebraical diviſion, and that quantity, which, multiplied into the diviſor, produces the dividend, is called the quotient: hence the quotient will always, in theſe caſes, be a negative quantity, ſince, if the diviſor be poſitive, it muſt be a negative quantity, that, multiplied into it, can produce the negative dividend; and if the diviſor be negative, it muſt be a negative quantity, that, multiplied into it, can produce the poſitive dividend: and it is evident that the operation muſt be conducted in theſe, by the ſame rules as in the former caſes.

EXAMPLES.

EXAMPLES.

Ex. $\frac{4a}{-a}=-4$; $\frac{-4a}{a}=-4$; $\frac{ab}{-a}=-b$;

$\frac{-ab}{a}=-b$; $\frac{x-y}{y-x}=-1$.

In the following examples a is ſuppoſed greater than b.

1ſt,
$$\begin{array}{rcl} \text{Neg.} & \text{Poſ.} & \text{Neg.} \\ -c\,) & -bc+ac & (\,b-a \\ & -bc & \\ \cline{2-2} & +ac & \\ & +ac & \\ \cline{2-2} & \cdot\,\cdot & \\ \cline{2-2} \end{array}$$

2d.
$$\begin{array}{rcl} \text{Poſ.} & \text{Neg.} & \text{Neg.} \\ c\,) & bc-ac & (\,b-a \\ & bc & \\ \cline{2-2} & -ac & \\ & -ac & \\ \cline{2-2} & \cdot\,\cdot & \\ \cline{2-2} \end{array}$$

3d.
$$\begin{array}{rcl} \text{Poſ.} & \text{Neg.} & \text{Neg.} \\ a-b\,) & -a^2+b^2 & (\,-a-b \\ & -a^2+ab & \\ \cline{2-2} & -ab+b^2 & \\ & -ab+b^2 & \\ \cline{2-2} & \cdot\,\cdot & \\ \cline{2-2} \end{array}$$

4th,
$$\begin{array}{rcl} \text{Neg.} & \text{Poſ.} & \text{Neg.} \\ -a+b\,) & a^2-b^2 & (\,-a-b \\ & a^2-ab & \\ \cline{2-2} & ab-b^2 & \\ & ab-b^2 & \\ \cline{2-2} & \cdot\,\cdot & \\ \cline{2-2} \end{array}$$

93. In all caſes of diviſion, the quotient, in its* abſolute magnitude, is the quotient of the abſolute magnitudes of the diviſor and dividend: if the diviſor and dividend be both poſitive, or both negative, this quotient is poſitive; but it is negative, if, of the diviſor and dividend, one be negative, and the other poſitive.

Let the abſolute magnitudes of the diviſor and dividend be A and nA, reſpectively;

then $\frac{+nA}{+A}$, or $\frac{-nA}{-A}$, is n; but $\frac{+nA}{-A}$, or $\frac{-nA}{+A}$, is $-n$.

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[*] Let a repreſent the ſum of the poſitive terms, and b the ſum of the negative terms in any expreſſion; then, by the abſolute magnitude of that quantity is meant $a \sim b$: Thus, of the quantity $3-7$, or -4, the abſolute magnitude is $7-3$, or, 4.

CHAP.

CHAPTER V. PART II.

OF FRACTIONS.

ARTICLE 94. IN chap. the firſt diviſion was defined the determining how many times the dividend contains the diviſor; under which idea, the dividend muſt be either equal to, or ſome exact multiple of the diviſor. But this definition was given merely with a view to the ſubſequent operation of diviſion, and is not to be conſidered as leading to an adequate idea of a quotient.

95. For the purpoſes of arithmetic, and algebra, it is neceſſary to admit any two numbers to be diviſor, and dividend, and to conſider the quotient as that part of the dividend, which the diviſor expreſſes.

Thus, let a and n be any two numbers whatſoever, $a \div n$, or, $\frac{a}{n}$, ſignifies the n^{th} part of a, and is called the quotient of a divided by n, becauſe it ſhews how many times, or parts of times, or times and parts of times n is contained in a.

96. When a remainder appears at the end of the operation of diviſion, the reſulting quotient is not, properly ſpeaking, the quotient of the two numbers employed, but of the diviſor and a number leſs than that employed as dividend, by the remainder.

Thus, in example 3. art. 24, the quotient, 10, is not the quotient of 83 divided by 8, but of (83 − 3, or) 80 divided by 8; and that remainder, 3, ſhews that 8 is not contained any exact whole number of times in 83; or, that an 8th part of 83 cannot be aſſigned in whole numbers.

If unit be ſubdivided into parts, an eighth part of this number may be aſſigned; for inſtance, if each unit be divided into eight equal parts, the number 83 will contain 83×8 ſuch parts; therefore $83 \times 8 \div 8$, or 83, is the number of eighths of an unit, that muſt be taken to make up an eighth part of the number 83; i. e. 83 eighths of an unit is the quotient of 83 divided by 8; or 8 is contained in 83 (not 83 times, but) 83 eighths of a time.

In a ſimilar manner it appears, that 8 is contained in 3 three eighths of a time; and ſince $83 = 80 + 3$, an eighth part of $83 =$ an eighth part of $80 +$ an eighth part of $3 = 10 +$ an eighth part of 3; or 8 is contained in 83, 10 times + three eighths of a time; and the quotient of 83 divided by 8, is $10 + \frac{3}{8}$, or $10\frac{3}{8}$, (for a quotient of this form is uſually annexed to the whole number without any ſign prefixed to it).

And in the ſame manner, in all other caſes, the remainder divided by the diviſor, muſt be added to the integral part of the quotient, in order to expreſs the true quotient.

97. In general, if $\frac{b}{n}$ be conſidered as expreſſing an n^{th} part of b, it will alſo expreſs the number of times, or parts of times, that n is contained in b; for becauſe 1 is contained in b, b times, n, being n times as great as 1, is contained in b an n^{th} part of b times. Alſo if unit be ſuppoſed divided into n equal parts, $\frac{b}{n}$ will expreſs b ſuch parts, or b n^{ths} of an unit; for an n^{th} part of b, is b times as great as an n^{th} of 1, therefore an n^{th} of $b = b$ n^{ths} of 1.

98. All quotients whatſoever, when expreſſed by placing the diviſor beneath the dividend, ſeparating them by a line, are called fractions; the dividend is called the numerator of the fraction, and the diviſor is called the denominator: for this reaſon, it ſeems,—that the diviſor denominates into what kind of parts unit may be ſuppoſed divided, and the dividend ſhews how many of ſuch parts are to be taken: Thus the quotient $\frac{b}{n}$ is a fraction, whatever be the values of b and n, whoſe numerator is b, and denominator n; and is conceived as repreſenting b n^{th} parts of an unit.

99. If the numerator of a fraction be leſs than the denominator, the value of the fraction is leſs than 1, as in the fractions $\frac{1}{2}$, $\frac{3}{7}$, $\frac{a}{a+n}$; for here, not ſo many parts are taken as the unit is ſuppoſed divided into, and conſequently not ſo much as an unit is taken. Such fractions are in common arithmetic called *proper*.

100. If the numerator be greater than the denominator, the value of the fraction is greater than 1, as in the fractions $\frac{3}{2}$, $\frac{5}{3}$, $\frac{a+b}{a}$; for here, more parts are take than the unit is ſuppoſed divided into, and conſequently more than an unit is taken.

101. If the numerator be equal to the denominator, the value of the fraction is unity.

Thus, $\frac{2}{2}$, $\frac{3}{3}$, $\frac{n}{n}$, are each equal to 1; for in this caſe exactly as many parts are taken, as unit is ſuppoſed divided into.

102. If the numerator be an exact multiple of the denominator, the value of the fraction is ſome whole number,—viz. that whole number which denominates the multiple.

Thus, $\frac{4}{1}=4$; $\frac{6}{2}=3$; $\frac{nb}{b}=n$. For if unit be divided into b parts, and b of thoſe parts be taken an unit is taken; if twice as much, or $2b$ of them be taken, 2 is taken; &c. if n times as much, or nb of them be taken, n is taken.

Fractions, under the deſcription of the three laſt articles are, in common arithmetic, called *improper*.

103. If the numerator of a fraction be multiplied by any number, (the denominator remaining the ſame,) the value of the fraction is multiplied by that number: for if double the number of parts be taken, twice as much is taken; if triple the number, three times as much is taken, &c.

Thus, $\frac{2\times 7}{3}$ or 14 3^{ds} $=7$ times 2 3^{ds}, or $\frac{2}{3}\times 7$.
$\frac{a\times b}{n}$ or ab n^{ths} $=b$ times a n^{ths} or $\frac{a}{n}\times b$.

104. Cor. Hence, becauſe ab n^{ths} is b times a n^{ths}, or a times b n^{th}, i. e. $\frac{ab}{n}$ is $\frac{a}{n}$ multiplied into b, or $\frac{b}{n}$ multiplied into a, it follows that $\frac{a}{n}\times b=\frac{b}{n}\times a$.

105. If the denominator of a fraction be multiplied by any number n, (the numerator remaining) the value of the fraction is divided by that number n, or becomes an n^{th} part of its former value: for if the number of parts into which unit is divided, be double in one caſe what it is in the other, the magnitude of each part is half the magnitude of a part in the other caſe; if the number of parts be treble, the magnitude of each part is a third, and ſo on: Therefore if the number of parts taken in two caſes be the ſame, i. e. if the numerators of two fractions be the ſame, the value of that fraction, whoſe denominator is double, or triple, &c. or n times the denominator of the other, will be half,

a third part, &c. or an n^{th} part, (in the reſpective caſes) of the value of that other fraction.

Thus, $\frac{2}{4\times 7}$ or $\frac{2}{28}$ is a ſeventh part of $\frac{2}{4}$ or is equal to $\frac{2}{4}\div 7$; $\frac{b}{mn}$ is an n^{th} part of $\frac{b}{m}$, or is equal to $\frac{b}{m}\div n$.

106. Cor. Hence, becauſe $\frac{b}{mn}$ is $\frac{b}{m}\div n$, or $\frac{b}{n}\div m$, it follows that $\frac{b}{m}\div n=\frac{b}{n}\div m$.

107. If any quantity be multiplied and divided by the ſame number, n, its value will not be altered; for n times an n^{th} part of any quantity is equal to the whole of that quantity.

108. Cor. Hence, and from Art. 103 and 105, it follows, that if the numerator and denominator of a fraction be multiplied by the ſame number, its value will not be altered.

109. Converſely to Art. 103, if the numerator of a fraction be divided by any number, the value of the fraction is divided by that number.

Thus, $\frac{14\div 7}{3}$ or $\frac{2}{3}$ is a ſeventh part of $\frac{14}{3}$; $\frac{nb\div n}{m}$, or $\frac{b}{m}$ is an n^{th} part of $\frac{nb}{m}$.—But unleſs the numerator be exactly equal to the dividing number, or to ſome multiple of it, the new numerator, which reſults, muſt be a fraction; for which reaſon the method of dividing fractions, given in Art. 105, is preferable.

110. Alſo, converſely to Art. 105, if the denominator of a fraction be divided by any number, the value of the fraction will be multiplied by that number.

Thus, $\frac{2}{28\div7}$ or $\frac{2}{4}$ is 7 times $\frac{2}{28}$; $\frac{a}{mn\div n}$, or $\frac{a}{m}$ is n times $\frac{a}{mn}$. But unleſs the denominator be exactly equal to the dividing number, or to ſome multiple of it, the method in Art. 103 is preferable.

111. Cor. If the numerator and denominator of a fraction be divided by the ſame quantity, its value will not be altered.

112. From Art. 108 and 111, it appears, that the ſame quotient may be expreſſed by innumerable different fractions:

$$\text{Ex. } \frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\&\text{c.};$$

$$\frac{ad}{bd}=\frac{a}{b}=\frac{ac}{bc}=\&\text{c.}$$

113. The definition of multiplication given in chap. 1ſt. extends to whole numbers only; but the word multiplication is uſed in a more general ſenſe to include all quantities whatſoever as multipliers; and the general definition is—"To multiply by any quantity, is, take the multiplicand, as many times, or parts of times, as there are units, or parts of units in the multiplier." Or, in other words,—"To multiply by any quantity, is to take that multiple, part, or parts of the multiplicand, which the multiplier expreſſes." Hence,

if the multiplier be any fraction $\frac{m}{n}$, ſince in $\frac{m}{n}$ there are m n^{ths} of an unit, to multiply any quantity by $\frac{m}{n}$, is to take m n^{ths} of that quantity.

Ex. 4 multiplied by $\frac{1}{2}$ is half 4, or 2; 5 multiplied by $\frac{2}{3}$ is 2 3^{ds} of 5; $\frac{a}{b} \times \frac{m}{n}$ is m n^{ths} of $\frac{a}{b}$.

114. The definition of an integral quotient,—viz. "That quantity, which, multiplied into the diviſor, produces the dividend," will, from this definition of multiplication be extended to all caſes of diviſion whatſoever; for let the diviſor and dividend, whether they be whole numbers or not, be repreſented by $\frac{a}{b}$ and $\frac{c}{d}$ reſpectively; then, if the quotient of $\frac{c}{d}$ divided by $\frac{a}{b}$ be $\frac{m}{n}$, i. e. if $\frac{a}{b}$ be contained in $\frac{c}{d}$ m n^{ths} of a time, $\frac{c}{d}$ is m n^{ths} of $\frac{a}{b}$, i.e. by the definition of multiplication, $\frac{c}{d} = \frac{a}{b} \times \frac{m}{n}$.

115. If two fractions have the ſame denominator, their ſum or difference is found by retaining the denominator, and ſubſtituting the ſum or difference of the numerators for a new numerator.

For, $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$, or, two 5^{ths} + one 5^{th} = three 5^{ths}; $\frac{2}{5} - \frac{1}{5} = \frac{1}{5}$, or, two 5^{ths} − one 5^{th} = one 5^{th}; $\frac{b}{m} \pm \frac{c}{m} = \frac{b \pm c}{m}$, i. e. $b\ m^{\text{ths}} + c\ m^{\text{ths}} = \overline{b+c}\ m^{\text{ths}}$; and $b\ m^{\text{ths}} - c\ m^{\text{ths}} = \overline{b-c}\ m^{\text{ths}}$.

116. Problem 1ſt. *To reduce an integral quantity to a fraction with a given denominator.*

* Rule. *Multiply the given quantity by the propoſed denominator; the product will be the numerator of the required fraction.*

Ex. Reduce 3 to a fraction, whoſe denominator is 5. $3 = \frac{3 \times 5}{5} = \frac{15}{5}$.

Ex. 2. Reduce b to a fraction, whoſe denominator is $c-d$. $b = \frac{b \times \overline{c-d}}{c-d} = \frac{bc-bd}{c-d}$.

The operation of the rule being to multiply and divide the given quantity by the ſame number, by art. 107, its value will not be changed.

117. An expreſſion conſiſting of an integral quantity and a fraction,—as $9 \frac{2}{5}$, or $a + \frac{b}{c}$, or $x - \frac{y}{2}$, is called a mixed quantity.

118. Problem 2d. *To reduce a mixed quantity to a fraction.*

Rule. *Reduce the integral part to a fraction, whoſe denominator is the denominator of the fractional part, by art.* 116; *incorporate the reſulting fraction with the given fraction, by art.* 115; *the reſult will be the fraction required.*

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* The truth of this rule appears in another light, by conſidering a whole number as a fraction, whoſe denominator is unity. For example:

$$3 = \frac{3}{1} = \frac{3 \times 5}{1 \times 5} = \frac{15}{5};$$

$$b = \frac{b}{1} = \frac{b \times \overline{c-d}}{1 \times \overline{c-d}} = \frac{bc-bd}{c-d} \quad \text{(See art. 108.)}$$

Ex. 1ſt. Reduce $7\frac{2}{3}$ to a fraction.

$7=\frac{7\times 3}{3}=\frac{21}{3}, \therefore 7+\frac{2}{3}=\frac{21}{3}+\frac{2}{3}=$ (art. 115.) $\frac{23}{3}$.

2d. Reduce $7-\frac{2}{3}$. $7-\frac{2}{3}=\frac{21}{3}-\frac{2}{3}=\frac{19}{3}$;

3d. Reduce $\frac{2}{3}-7$. $\frac{2}{3}-7=\frac{2}{3}-\frac{21}{3}=\frac{-19}{3}$;

4th. Reduce $a+\frac{dx}{b-c}$. $a+\frac{dx}{b-c}=\frac{ab-ac}{b-c}+\frac{dx}{b-c}$ $=\frac{ab-ac+dx}{b-c}$;

5th. Reduce $a-b+\frac{cy-ad}{x}$. $a-b+\frac{cy-ad}{x}$ $=\frac{ax-bx}{x}+\frac{cy-ad}{x}=\frac{ax-bx+cy-ad}{x}$:

6th. Reduce $a-b-\frac{cy-ad}{x}$. $a-b-\frac{cy-ad}{x}=$ $\frac{ax-bx}{x}-\frac{cy-ad}{x}=\frac{ax-bx-\overline{cy-ad}}{x}=\frac{ax-bx-cy+ad}{x}$

119. Problem 3d. *To reduce a fraction (where poſſible) to an integral, or mixed quanity.*

Rule. *Divide the numerator by the denominator, and the integral part of the quotient* $+\frac{remainder}{denominator}$ *will be the value of the given fraction in the required form.*

Ex. 1. $\frac{23}{3}=7\frac{2}{3}$; 2d. $\frac{54}{9}=6$;

3d. $\frac{ab-c}{b}=a-\frac{c}{b}$; 4th. $\frac{mn}{n}=m$.

120. A fraction, whoſe numerator and denominator have a common meaſure, (or number, which divides each of them without a remainder,) greater than unity, may be reduced to an equivalent fraction in lower terms, by dividing both the numerator and denominator by this common meaſure. (See art. 111.)

Thus, $\frac{45}{120}$ may be reduced to $\frac{9}{24}$ by dividing both the numerator and denominator by 5, and $\frac{9}{24}$ may be reduced to $\frac{3}{8}$ by dividing its numerator and denominator by 3.

In the ſame manner $\frac{6}{12}$ may be reduced to $\frac{3}{6}$, which may be reduced to $\frac{1}{2}$. And the greater the common meaſure, which is employed, the leſs will be the terms of the reſulting fraction; if the greateſt common meaſure, which the terms admit of, be employed, the fraction will be reduced to its loweſt terms.

121. Problem 4th. *To find the greateſt common meaſure of two numbers.*

Rule. *Divide the greater number by the leſs, the leſs by the remainder, that remainder by the new remainder, and ſo on, continually dividing the preceding diviſor by the remainder till nothing remain, and the laſt diviſor will be the greateſt common meaſure of the given numbers.*

Let a and b be the two numbers, and let a contain b, p times, with a remainder c; let b contain c, q times, with a remainder d; let c contain d, r times, with a remainder e; and ſo on, till nothing remain,—the laſt diviſor will be the greateſt common meaſure of a and b.

$b)\,\underline{a}\,(p$
$c)\,\underline{b}\,(q$
$d)\,\underline{c}\,(r$
$e)\,d\,(s$

To demonſtrate this, it muſt be obſerved,

1ſt. That if one quantity meaſure another, it will meaſure any multiple of that other; for, if x be contained in a, n times, it will be contained in ac, nc times.

2d. That if a quantity meaſure two others, it will meaſure their ſum and their difference; for if x be contained in a, m times, and in b, n times, it will be contained in $\overline{a+b}$, $\overline{m+n}$ times; and in $\overline{a\sim b}$, $\overline{m\sim n}$ times. Now in the operation above, if e be the laſt diviſor, e meaſures d, being contained in it exactly s times; therefore (by Obſ. 1ſt.) e meaſures rd; therefore (by Obſ. 2d.) it meaſures $\overline{rd+e}$, or c;* therefore it meaſures qc; therefore it meaſures $\overline{qc+d}$, or b; therefore it meaſures pb, and $\overline{pb+c}$, or a.

e then is found to be a common meaſure of a and b; it is alſo their greateſt common meaſure; for whatever quantity meaſures a and b, muſt meaſure $\overline{a-pb}$, or c; therefore it muſt meaſure qc and $\overline{b-qc}$ or d; therefore it muſt meaſure rd, and $\overline{c-rd}$ or e;

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* A dividend always is equal to the *integral quotient* × the *diviſor*, together with the *remainder*.

a quantity, therefore, greater than e, as $e+m$, if it could meaſure a and b, muſt meaſure e, which is abſurd; therefore e is the greateſt common meaſure. It is evident that the demonſtration does not at all depend on the number of diviſions. b might have been ſuppoſed the laſt diviſor, or c, or d, or any remainder after the diviſion by e.

Ex. To find the greateſt common meaſure of 189 and 1204.

$$
\begin{array}{lllll}
189\ )\ \overset{*}{1}204\ (\ 6 & & & & \\
1134 & & & & \\
\hline
 & 70\ )\ 189\ (\ 2 & & & \\
 & 140 & & & \\
\hline
 & & 49\ )\ 70\ (\ 1 & & \\
 & & 49 & & \\
\hline
 & & & 21\ (\ 49\ (\ 2 & \\
 & & & 42 & \\
\hline
 & & & & 7\ )\ 21\ (3 \\
 & & & & 21 \\
\hline
 & & & & \cdot\ \cdot \\
\hline
\end{array}
$$

By proceeding according to the rule, 7 is found to be the greateſt common meaſure of the given numbers, 189 and 1204.

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* Each diviſor will neceſſarily be leſs than the preceding by unity at leaſt; therefore the operation will always come to a concluſion; for after ſome number of diviſions, (unleſs the operation terminate ſooner) the remainder will be 1, in which caſe the operation will conclude with the next diviſion; ſince whatever number it be, that is divided by 1, there can be no remainder.

Hence, $\frac{189}{1204}$, reduced to its loweſt terms, is

$$\frac{189 \div 7}{1204 \div 7} = \frac{27}{172}.$$

The greateſt common meaſure of 45 and 63, after the ſame manner is found to be 9; therefore $\frac{45}{63}$ reduced to its loweſt terms is $\frac{45 \div 9}{63 \div 9} = \frac{5}{7}$.

122. Problem 5th. *To find the greateſt common meaſure of three quantities, a, b, and c.*

Rule. *Find the greateſt common meaſure (m) of any two, as of a and b: and alſo the greateſt common meaſure (n) of either of theſe two, and of the third, as of b and c; the greateſt common meaſure of m and n is the greateſt common meaſure of a, b, and c.*

For in the demonſtration of the laſt article, it appears, that every common meaſure of *a* and *b* is a meaſure of *m*; and that every common meaſure of *b* and *c* is a meaſure of *n*; therefore *every* common meaſure of *a*, *b*, and *c*, being a meaſure of *m*, becauſe a common meaſure of *a* and *b*, and a meaſure of *n*, becauſe a common meaſure of *b* and *c*, muſt be a common meaſure of *m* and *n*; therefore the *greateſt* common meaſure of *a*, *b*, and *c* is a common meaſure of *m* and *n*: again, by the ſame article it appears, that *every* meaſure of *m*, is a common meaſure of *a* and *b*, and *every* meaſure of *n*, a common meaſure of *b* and *c*; therefore *every* common meaſure of *m* and *n* muſt muſt be a common meaſure of *a*, *b*, and *c*; therefore the *greateſt* common meaſure of *m* and *n* is a common meaſure of *a*, *b*, and *c*: ſince then the greateſt common meaſure of *a*, *b*, and *c* muſt be *one* of the common meaſures of *m* and *n*, and that the *greateſt* common meaſure of *m* and *n* is a common meaſure of *a*, *b*, and *c*, it follows that the greateſt common meaſure of *m* and *n* muſt be the greateſt common meaſure of *a*, *b*, and *c*.

123. Problem 6th. *To find the greateſt common meaſure of four quantities, a, b, c, and d.*

Rule. *Find the greateſt common meaſure (m) of any two of them, and the greateſt common meaſure (n) of the other two. The greateſt common meaſure of m and n is the greateſt common meaſure required.*

124. If there be any even number of quantities and *m* be the greateſt common meaſure of two of them; *n*, of any other two; *l*, of any other two; &c. the greateſt common meaſure of *m*, *n*, *l*, &c. will be the greateſt common meaſure of the given quantities.

125. If there be any odd number of quantities, and *m* be the greateſt common meaſure of two of them; *n* of other two; &c. and *t* of the remaining odd one, and any one of the others, the greateſt common meaſure of *m*, *n*, &c. and *t* will

be

be the greateſt common meaſure of the given quantities.

The demonſtration of the three laſt articles is exactly ſimilar to that of art. 122.

126. Problem 7th. *To reduce fractions to others of equal value, having a common denominator.*

Rule. *Take for new numerators, the product of the numerator of each fraction, multiplied into the denominators of all the reſt; and the product of all the denominators, for a common denominator.*

Ex. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ to fractions with a common denominator. $\frac{1.3.4}{2.3.4}$, $\frac{2.2.4}{3.2.4}$, and $\frac{3.2.3}{4.2.3}$ $= \frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$, are the fractions required.

Ex. 2. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$ to fractions with a common denominator, $\frac{adn}{bdn}$, $\frac{cbn}{bdn}$, and $\frac{bdm}{bdn}$ are the fractions required.

Ex. 3. Reduce $\frac{2}{3}$ and $3\frac{1}{2}$ to fractions with a common denominator. $3\frac{1}{2} = \frac{7}{2}$. $\frac{2}{3}$ and $\frac{7}{2}$ reduced to a common denominator, are $\frac{4}{6}$ and $\frac{21}{6}$.

Ex. 4th. Reduce a, $\frac{bx}{c-d}$ and $\frac{cx}{d}$ to fractions with a common denominator.

$$\frac{a}{1}, \frac{bx}{c-d} \text{ and } \frac{cx}{d} = \frac{a \times \overline{c-d} \times d}{1 \times \overline{c-d} \times d}, \quad \frac{1 \times bx \times d}{1 \times \overline{c-d} \times d}$$

and $\frac{1 \times cx \times \overline{c-d}}{1 \times \overline{c-d} \times d}$ reſpectively, $=$

$\frac{acd-ad^2}{cd-d^2}$, $\frac{bdx}{cd-d^2}$ and $\frac{c^2x-cdx}{cd-d^2}$ reſpectively.

By the operation of the rule, the numerator and denominator of each fraction are multiplied into the ſame quantity; therefore by art. 108, the values of the fractions will not be changed.

127. Fractions may be reduced to a common numerator, by taking for new denominators, the denominator of each fraction, multiplied into the numerators of all the other fractions, and the product of all the numerators for a common numerator.

Thus, $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are equal to $\frac{ace}{bce}$, $\frac{ace}{dae}$, and $\frac{ace}{fac}$ reſpectively.

128. If, of certain fractions, $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, &c., b, the greateſt denominator, be ſome multiple of each of

the other denominators, as the m multiple of d, the n multiple of f, &c. i. e. if $b = md = nf =$ &c. the fractions may be reduced to a common denominator, b, by multiplying the terms of the fractions $\frac{c}{d}$ into m, the terms of the fraction $\frac{e}{f}$ into n, &c. for they will thus become $\frac{a}{b}$, $\frac{mc}{md}$, $\frac{nc}{nf}$, &c. or $\frac{a}{b}$, $\frac{mc}{b}$, $\frac{nc}{b}$, &c.

Ex. 1. Reduce $\frac{7}{12}$ and $\frac{2}{3}$ to a common denominator. $12 = 3 \times 4$; therefore the fractions are $\frac{7}{12}$ and $\frac{2 \times 4}{3 \times 4}$; or $\frac{7}{12}$ and $\frac{8}{12}$.

Ex. 2. Reduce $\frac{ab}{mx}$ and $\frac{d}{x}$ to fractions with a common denominator. $mx = x \times m$; therefore the fractions are $\frac{ab}{mx}$ and $\frac{md}{mx}$.

129. If a common factor, or common factors be found in two or more of the denominators, the fractions may be reduced to a common denominator, by taking, for this common denominator, the product of all the *different* factors, that are to be found among the denominators, and multiplying the numerator of each fraction by the product of the ſame factors, as were introduced into its denominator, in order to reduce it to the common denominator.

Note. If the ſame factor be *repeated* in any denominator, it muſt be as often repeated in the common denominator.

Ex. 1. $\frac{7}{20}$ and $\frac{1}{12}$, or $\frac{7}{5.4}$ and $\frac{1}{3.4}$ may be reduced to a common denominator, 3. 4. 5. by multiplying the terms of the former fraction by 3, and the terms of the latter by 5: For ſo they become $\frac{7\times3}{20\times3}$ and $\frac{1\times5}{12\times5} = \frac{21}{60}$ and $\frac{5}{60}$ reſpectively.

Ex. 2. $\frac{a^2}{mn}$, $\frac{b}{n}$, and $\frac{c}{m}$, may be reduced to a common denominator, *mn*, by multiplying the terms of $\frac{b}{n}$ into *m*, and the terms of $\frac{c}{m}$ into *n:* for ſo the fractions become $\frac{a^2}{mn}$, $\frac{bm}{mn}$, and $\frac{cn}{mn}$ reſpectively.

Ex. 3. $\frac{a^2x^2}{dy^2}$, $\frac{x^6}{dy^2c^2}$, and $\frac{y^3}{ce}$ may be reduced to a common denominator, dy^2c^2e, by multiplying the terms of $\frac{a^2x^2}{dy^2}$ into c^2e, and the terms of $\frac{x^6}{dy^2c^2}$ into e, and the terms of $\frac{y^3}{ce}$ into dy^2c: For ſo they become $\frac{a^2x^2c^2e}{dy^2c^2e}$, $\frac{x^6e}{dy^2c^2e}$, and $\frac{dy^5c}{dy^2c^2e}$ reſpectively.

The uſe of the two laſt articles is to obtain lower terms than what would reſult from the rule in art. 126.

130. **Problem 8th.** *To reduce two or more fractions to a common denomination in the loweſt terms.*

Rule. *Reduce the fractions to a common denominator by art.* 126, *and divide the terms of the reduced fractions by the greateſt common meaſure of this denominator and all the numerators.*

For, if the fractions, when reduced to a common denominator, be $\frac{a}{b}$, $\frac{c}{b}$, $\frac{d}{b}$, &c. the terms muſt be divided by ſome common meaſure of a, b, c, d, &c. in order that the reſulting fractions may be equal to the former reſpectively, and alſo have a common denominator: any common meaſure of a, b, c, d, &c. will produce fractions of ſuch a deſcription, and the greateſt common meaſure will produce them in the leaſt poſſible terms.

Ex. Reduce $\frac{1}{12}$ and $\frac{7}{8}$ to fractions in the loweſt terms with a common denominator.

$\frac{1}{12}$ and $\frac{7}{8}$ are reduced by article 126, to $\frac{8}{96}$ and $\frac{84}{96}$; The greateſt common meaſure of 8, 96, and 84 is 4; therefore the required fractions are

$$\left(\frac{8\div 4}{96\div 4}, \text{ and } \frac{84\div 4}{96\div 4}, \text{ or}\right)\ \frac{2}{24} \text{ and } \frac{21}{24}.$$

ADDITION

ADDITION OF FRACTIONS.

131. Rule. *If fractions have a common denominator, take the ſum of the numerators, and ſubjoin the common denominator*, (Art. 115.)

Ex. $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$. $\frac{a}{b}+\frac{c-d}{b}=\frac{a+c-d}{b}$.

If the fractions have not a common denominator, firſt reduce them to others that have, and then proceed as above.

Ex. 1ſt. $\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}$.

Ex. 2d. $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd.}$.

Ex. 3d. $1\frac{3}{7}+\frac{5}{14}=\frac{10}{7}+\frac{5}{14}=\frac{20}{14}+\frac{5}{14}=\frac{25}{14}$.

Ex. 4th. $a+\frac{b^2}{c}=\frac{ac}{c}+\frac{b^2}{c}=\frac{ac+b^2}{c}$.

Ex. 5th. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=\frac{6x}{12}+\frac{4x}{12}+\frac{3x}{12}=\frac{13x}{12}$.

SUBTRACTION OF FRACTIONS.

132. Rule. *If the fractions have a common denominator, ſubtract the numerator of the ſubtrahend from the numerator of the minuend, for a new numerator, and ſubjoin the common denominator.*

Ex. $\frac{4}{5}-\frac{3}{5}=\frac{1}{5}$. $\frac{3}{5}-\frac{4}{5}=-\frac{1}{5}$. $\frac{a}{b}-\frac{c-d}{b}$
$=-\frac{a-\overline{c-d}}{b}=\frac{a-c+d}{b}$. $2\frac{2}{3}-1\frac{1}{3}=1\frac{1}{3}$.
$1\frac{2}{3}-2\frac{1}{3}=-1+\frac{1}{3}$.

If the fractions have not a common denominator, firſt reduce them to others that have, and then proceed as above.

Ex. $\frac{3}{4}-\frac{2}{5}=\frac{15}{20}-\frac{8}{20}=\frac{7}{20}$.

$\frac{2}{5}-\frac{3}{4}=\frac{8}{20}-\frac{15}{20}=-\frac{7}{20}$.

$\frac{a}{b}-\frac{c-d}{d}=\frac{ad}{bd}-\frac{cb-bd}{bd}=\frac{ad-\overline{cb-bd}}{bd}$
$=\frac{ad-cb+bd}{bd}$.

MULTIPLICATION

MULTIPLICATION OF FRACTIONS.

133. Any quantity is multiplied by a fraction, by multiplying it into the numerator, and dividing it by the denominator.

For, let the quantity be c, and the multiplier $\frac{m}{n}$. One n^{th} of c is $\frac{c}{n}$; therefore m n^{ths} of c, or, (by the definition of multiplication) $c\times\frac{m}{n}$, is m times $\frac{c}{n}$, $=$ (art. 103.) $\frac{cm}{n}$.

If $c=\frac{a}{b}$, c multiplied by $\frac{m}{n}=\frac{a}{b}$ multiplied by $\frac{m}{n}=\frac{a}{b}\div n\times m=$ (art. 105.) $\frac{a}{bn}\times m=$ (art. 103.) $\frac{am}{bn}$. Hence the following

134. Rule. For Multiplication of Fractions.

Multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Ex. $\frac{2}{3}\times\frac{3}{5}=\frac{6}{15}$. $\quad\frac{a}{b}\times\frac{c-d}{x}=\frac{ac-ad}{bx}$.

$2\times\frac{4}{5}$, or $\frac{2}{1}\times\frac{4}{5}=\frac{8}{5}$. $\quad -m\times\frac{ab}{c}=-\frac{mab}{c}$.

135. Hence the property, which was proved of integral factors in art. 70. obtains alſo in fractional factors,—viz. that the multiplicand × multiplier = multiplier × multiplicand.

For it appears by the above, that $\frac{a}{b}$ multiplied by $\frac{c}{d}$ is $\frac{a \times c}{bd}$ and that $\frac{c}{d}$ multiplied by $\frac{a}{b}$ is $\frac{c \times a}{db} = \frac{a \times c}{bd}$.

136. Cor. If the product of two quantities be divided by a third, the reſult is the ſame, as if either of the factors had been divided by that third quantity, and the quotient multiplied into the other factor.

For $\frac{mc}{n}$, which is mc divided by n, $= \frac{m}{n} \times c$, which is the quotient of m divided by n, multiplied into c.

137. The general rule for multiplication in art. 84, may be extended to fractional expreſſions.

For the product reſulting, in any caſe from the application of that rule, if it be reduced to a ſingle fraction, will evidently be found the ſame as the product obtained, when the multiplier and multiplicand, having been firſt reduced to ſingle fractions, are multiplied together, according to the rule in art. 134.

Ex. 1ſt. $\overline{\frac{a}{b}-\frac{c}{d}}\times\frac{e}{f}=\frac{ae}{bf}-\frac{ce}{df}=$ (art. 129.) $\frac{aed-bce}{bdf}$.

For, $\frac{a}{b}-\frac{c}{d}=$ (art. 132.) $\frac{ad-cb}{bd}$, and $\frac{ad-cb}{bd}\times\frac{e}{f}$ is, by article 134, $\frac{ade-cbe}{bdf}$.

Ex. 2d. $\overline{x-\frac{a^2}{y}}\times\overline{c-\frac{cd}{a}}=cx-\frac{ca^2}{y}-\frac{cdx}{a}+\frac{cda^2}{ay}$ $=\frac{acxy-ca^2-cdxy+cda^2}{ay}$, the product of $\frac{xy-a^2}{y}$ multiplied into $\frac{ca-cd}{a}$, determined by art. 134.

Ex. 3d. $\overline{\frac{2x^2}{b}-\frac{3a^2}{b}}\times\overline{\frac{2x^2}{c}+\frac{3a^2}{c}}$

$=\frac{4x^4}{bc}-\frac{6a^2x^2}{bc}+\frac{6a^2x^2}{bc}-\frac{9a^4}{bc}$

$=\frac{4x^4}{bc}-\frac{9a^4}{bc}=\frac{4a^4-9a^4}{bc}$.

$=\frac{2x^2-3a^2}{b}\times\frac{2x^2+3a^2}{c}$.

DIVISION

DIVISION OF FRACTIONS.

138. Any quantity c is divided by a fraction $\frac{m}{n}$, by multiplying it into the denominator, and dividing it by the numerator.*

For m is contained in c an m^{th} of c times; therefore $\frac{m}{n}$ (which, being only an n^{th} of m, is contained in c, n times as often as m is) is contained in c, n times an m^{th} of c times, i. e. quotient of c divided by $\frac{m}{n}$ is n times an m^{th} of c, or n times, $\frac{c}{m}$ = (art. 103.) $\frac{cn}{m}$ or $c \times \frac{n}{m}$.

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* If unit be divided into $\frac{m}{n}$ parts, i. e. if m n^{ths} of a part conſtitute an unit, m parts will conſtitute n units, and one part will conſtitute an m^{th} of n units, therefore b parts will conſtitute b m^{ths} of n units, or $\frac{b}{m} \times n$ units; therefore $\frac{b}{\frac{m}{n}} = \frac{b}{m} \times n = b \times \frac{n}{m}$.

Now if the quotient of b divided by $\frac{m}{n}$, be $\frac{bn}{m}$ the quotient of $\frac{b}{d}$ (which is only a d^{th} part of b) divided by the ſame quantity, $\frac{m}{n}$ muſt be a d^{th} part of $\frac{bn}{m}$, or $\frac{bn}{dm} = \frac{b}{d} \times \frac{n}{m}$.

If $c = \frac{a}{b}$, $c \div \frac{m}{n} = \frac{a}{b} \div \frac{m}{n} = \frac{a}{bm} \times n = \frac{an}{bm}$.

Aliter. Let $\frac{a}{b} = x$ and $\frac{m}{n} = y$; then multiplying by b, $a = bx$, and multiplying by n, $m = ny$; alſo $an = nbx$ and $bm = nby$; $\therefore \frac{nbx}{nby} =$ (art. 111.) $\frac{x}{y}$ = (by ſuppoſition) $\frac{a}{b} \div \frac{m}{n} = \frac{an}{bm} = \frac{a}{b} \times \frac{n}{m}$. And if $\frac{a}{b} = c$, it becomes $c \div \frac{m}{n} = \frac{cn}{m} = c \times \frac{n}{m}$.

Hence the following

Rule for Diviſion of Fractions. *Multiply by the inverted diviſor.*

Ex. 1. $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$.

2. $\frac{a}{b} \div \frac{c-d}{x} = \frac{a}{b} \times \frac{x}{c-d} = \frac{ax}{bc-bd}$.

3. $5 \div \frac{1}{3} = 5 \times \frac{3}{1} = 15$.

4. $a \div \frac{bc}{x} = a \times \frac{x}{bc} = \frac{ax}{bc}$.

5. $m \div \overline{\frac{a}{b} - \frac{c}{d}} = m \div \frac{ad-bc}{bd} = m \times \frac{bd}{ad-bc}$.

140. Unity divided by any quantity, is called the reciprocal of that quantity.

Thus, $\frac{1}{a}$ is the reciprocal of a.

$\frac{1}{\frac{b}{a}}$ or (art. 139.) $1 \times \frac{a}{b} = \frac{a}{b}$ is the reciprocal of $\frac{b}{a}$.

The reciprocal of $\overline{\frac{a}{b} \div \frac{c}{d}}$ is $\frac{1}{\frac{a}{b} \div \frac{c}{d}}$ = (art. 139.)

$\frac{1}{ad \div bc}$ = (art. 110.) $\frac{1}{ad} \times bc = \frac{bc}{ad}$.

141. Cor. The reciprocal of any quantity, is that quantity with its numerator and denominator inverted.

For, $\frac{1}{a}$ is a, or $\frac{a}{1}$, inverted. $\frac{a}{b}$ is $\frac{b}{a}$ inverted.

$\frac{bc}{ad} = \frac{c}{d} \div \frac{a}{b}$, is $\frac{a}{b} \div \frac{c}{d}$, inverted.

142. Cor. 2. If m be the reciprocal of n, n is the reciprocal of m.

143. Cor. 3. Diviſion is multiplication by the reciprocal.

For $m \div n$ is $m \times \frac{1}{n}$. $\frac{a}{b} \div \frac{c}{d}$ is $\frac{a}{b} \times \frac{d}{c}$.

144.

144. Cor. 4. Multiplication is diviſion by the reciprocal.

For $\frac{a}{b} \times \frac{c}{d}$ is $\frac{a}{b} \div \frac{d}{c}$.

145. Any fraction may be conſidered as ariſing entirely from the multiplication of factors.

Thus, $\frac{abc}{def} = a \times b \times c \times \frac{1}{d} \times \frac{1}{e} \times \frac{1}{f}$, is the product of the factors a, b, c, $\frac{1}{d}$, $\frac{1}{e}$ and $\frac{1}{f}$.

146. The value of any expreſſion obtained by multiplication, or by diviſion, or by both, is not affected by the order, in which the multiplication and diviſion be carried on.

Thus, $\frac{abc}{def} = \frac{a}{d} \times \frac{b}{e} \times \frac{c}{f} = abc \times \frac{1}{def} = ab \times \frac{c}{def}$ $= ac \times \frac{1}{d} \times \frac{b}{ef} = a \div def \times bc =$ &c.

In the operation of diviſion, when the diviſor is the product of factors, we may divide by the factors ſucceſſively.

Thus, in dividing 210 by 35, becauſe $35 = 5 \times 7$, we may firſt divide by 7, and then divide the quotient (30) by 5. The laſt quotient, which is 6, is the quotient required.

For, $\frac{210}{5 \times 7} = \frac{210}{7} \div 5 = 30 \div 5$.

SCHOLIUM.

147. Becauſe any negative quantity, $-a$, divided by a poſitive quantity, b, gives the ſame quotient as $+a$, divided by $-b$, viz. the ſame quotient, in abſolute magnitude, as $+a$ divided by $+b$, but with a negative ſign, and becauſe a fraction repreſents the quotient reſulting from the diviſion of the numerator by the denominator, it is indifferent, in caſes where either the numerator or denominator of a fraction is negative, whether the negative ſign be prefixed to the numerator, or to the denominator, or to the whole fraction.

Thus, $\frac{-10}{2}$, $\frac{10}{-2}$ and $-\frac{10}{2}$ are equivalent expreſſions; being each equal to -5.

Note. The negative ſign is, in theſe caſes, uſually prefixed to the whole fraction.

If the ſign of the numerator, or of the denominator, be changed, the ſign of the quotient is changed. If the ſign of both numerator and denominator be changed, the ſign of the quotient remains unaltered. Hence, if we change the ſign of the numerator, or of the denominator, and wiſh to preſerve the value of the fraction, we muſt change the ſign, that is prefixed (or underſtood to be prefixed) to the whole of the fraction, into the contrary ſign.

Ex. $\frac{a}{b}$ (or $+\frac{a}{b}$) $=\frac{-a}{-b}$ (or $+\frac{-a}{-b}$) $=-\frac{-a}{b}$ $=-\frac{a}{-b}$. $-\frac{a}{b}=-\frac{-a}{-b}=\frac{-a}{b}$ (or $+\frac{-a}{b}$) $=\frac{a}{-b}$ (or $+\frac{a}{-b}$.)

$\frac{x^2-y^2}{a}$ (which is *poſitive*, if x^2 be greater than y^2, but *negative*, if y^2 be greater than x^2) $=\frac{y^2-x^2}{-a}=-\frac{y^2-x^2}{a}=-\frac{x^2-y^2}{-a}$.

OF DECIMAL FRACTIONS.

148. The variety of denominators in the arithmetic of vulgar fractions, renders it tedious and troubleſome: for when numbers are divided into different kinds of parts, they cannot be eaſily compared.

This conſideration gave riſe to the invention of decimal fractions, in which the unit is divided into like parts, and the diviſions, and ſubdiviſions are regulated by the ſame ſcale, as is uſed in the arithmetic of integers.

149. Decimal fractions are thoſe, whoſe denominators are 10, or ſome * power of 10; as 100, 1000, 10000, &c.

Theſe denominators are not expreſſed, but implied by a point or comma prefixed to the left hand of the number; the denominator being always underſtood to conſiſt of unity, with as many cyphers, as there are figures after the decimal point; if therefore the number of places in the numerator be leſs than the number of cyphers in the intended denominator, as many cyphers are interpoſed between the numerator and the decimal point, as may be neceſſary to complete that number.

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* For definition of powers, ſee firſt art. in ch. on Involution.

Thus, $\frac{3}{10}$ is written .3; $2+\frac{3}{10}$ is written 2.3;

$\frac{3}{100}$ is written .03; $2+\frac{3}{100}$ is written 2.03; &c.

$\frac{24}{100}$ is written .24; $\frac{24}{1000}$ is written .024;

$5+\frac{24}{100}$ is written 5.24; &c. &c.

150. In any decimal expreſſion, the firſt figure after the point may be conſidered as expreſſing ſo many tenths; the ſecond figure, as expreſſing ſo many hundredths; the third figure, as expreſſing ſo many thouſandths; and ſo on.

Ex. .23546, which ſignifies $\frac{23546}{100000}$, if the numerator be divided into its component parts, becomes

$$\frac{20000}{100000}+\frac{3000}{100000}+\frac{500}{100000}+\frac{40}{100000}+\frac{6}{100000}$$

$=$ (art. 111, and obſ. 3, art. 24.) $\frac{2}{10}+\frac{3}{100}+\frac{5}{1000}+\frac{4}{10000}+\frac{6}{100000}$, or, $2\ 10^{\text{ths}}+3\ 100^{\text{ths}}+5\ 1000^{\text{ths}}+4\ 10000^{\text{ths}}+6\ 100000^{\text{ths}}$.

Ex. 2. .037, which ſignifies $\frac{37}{1000}$, is equal to

$$\frac{000}{1000}+\frac{30}{1000}+\frac{7}{1000}=\frac{3}{100}+\frac{7}{1000}.$$

151. Cor. 1. Hence, as in integers, ſo alſo in decimals, the value of any figure is decreaſed ten fold by being removed one place to the right. And as the value of any figure in the firſt place of decimals is a 10th part of the value of the ſame figure in the place of units, it follows, that in any expreſſion conſiſting of a whole number with a decimal annexed, the value of each figure, from the higheſt place of integers, to the loweſt place of decimals, is regulated by the ſame ſcale; which renders theſe fractions peculiarly convenient for the operations of arithmetic.

152. Cor. 2. The addition of cyphers to the right hand of a decimal does not alter its value.

For as many cyphers as are added to the numerator, ſo many more places does it conſiſt of; therefore ſo many more cyphers are added to the denominator; now the addition of the ſame number of cyphers to the numerator and denominator of a fraction does not alter its value.

Thus, .2, .20, .200, &c. or $\frac{2}{10}$, $\frac{20}{100}$, $\frac{200}{1000}$, &c. are all equal to each other.

153. Cor. 3. If the decimal point be removed one place to the left, the number is divided by 10; if it be removed two places, the number is divided by 100; &c. And converſely if the decimal point be removed one place to the right,

the number is multiplied by 10; if it be removed two places, the number is multiplied by 100; &c.

For, if the decimal point be removed one place to the left, the figure which did repreſent units will become 10^{ths}; the figure which did repreſent tens, will become units; &c.; the figure which did repreſent 10^{ths}, will become 100^{ths}; &c. i. e. every part of the given number will be divided by 10; therefore the whole given number will be divided by 10. And in the ſame manner it may be ſhewn, that if the decimal point be removed two places to the left, the number will be divided by 100, and ſo on. And that if the decimal point be removed to the right, 1, 2, 3, &c. places, the number will be multiplied by 10, 100, 1000, &c. in the reſpective caſes.

Ex. Let the given number be 37.092, or (placing cyphers indefinitely to the right and left) &c. 00037.092000, &c.

Then $3.7092 = 37.092 \div 10$; $0.37092 = 37.092 \div 100$; $0.037092 = 37.092 \div 1000$; &c.

$370.92 = 37.092 \times 10$; $3709.2 = 37.092 \times 100$; $37092.0 = 37.092 \times 1000$; $370920.0 = 37.092 \times 1000$; &c.

154. Cor. 4. Hence, if from any decimal, or integral number, and decimal annexed, the decimal point be taken away, the number is multiplied by the denominator of the decimal.

Ex. 1ſt. If from 2.5 the decimal point be taken away, it becomes $25 = 2.5 \times 10$.

Ex. 2d. If from 3.01 the decimal point be taken away, it becomes $301 = 3.01 \times 100$.

155. Cor. 5. Hence, any mixed number, conſiſting of an integral number and a decimal, is equal to a fraction whoſe denominator is the denominator of the decimal, and whoſe numerator conſiſts of all the figures of the given mixed number.

Thus, $2.7 = \frac{27}{10}$. $35.07 = \frac{3507}{100}$.

For by the laſt art. it appears, that the ſubtraction of the decimal point multiplies the number by the denominator of the decimal; therefore to reduce the number to its former value, it muſt be divided by the denominator.

156. Cor. 6. Decimals may be reduced to a common denominator, by adding cyphers to the right (where neceſſary) till the number of decimal places be the ſame in all.

Ex. .5, .77, .004, and .2354, reduced to a common denominator, are .5000, .7700, .0040, and .2354 reſpectively.

ADDITION

ADDITION OF DECIMALS.

157. **Rule.** *Place the given numbers, tenths under tenths, hundredths under hundredths, &c. units under units, tens under tens, &c. and proceed as in addition of integers; carrying one for every ten in the ſum of any row, towards the ſum of the next; and ſetting down the* exceſs *above ten or tens for a figure of the ſame claſs as the row from which it was obtained.*

EXAMPLES.

1ſt. .56
·79
Sum 1·35

2d. .88
.706
·954
Sum 2.540

3d. 74·369
2.711
Sum 77·080

4th. 11.05
2.65
6.3
Sum 20.00

Ten in any place of decimals, or integers, being equal to one in the next higher place, the reaſon of the rule is manifeſt.

The *decimal* cyphers on the right hand of all the ſignificant figures in the ſum obtained may be omitted.

Thus, in Ex. 2d, 3d, and 4th, the ſums might have been written

2.54, 77.08, and 20 reſpectively.

SUBTRACTION OF DECIMALS.

158. Rule. *Place the ſubtrahend under the minuend, tenths under tenths, &c. as in addition; and proceed as in ſubtraction of integers; making each figure of the remainder of the ſame rank as the figures of the ſubtrahend and minuend from which it was obtained.*

Ex.

$$\begin{array}{r} .72 \\ .34 \\ \hline .38 \\ \hline \end{array} \qquad \begin{array}{r} 3.15 \\ 1.25 \\ \hline 1.9 \\ \hline \end{array} \qquad \begin{array}{r} 1.2 \\ .46 \\ \hline .74 \\ \hline \end{array}$$

The ſame reaſoning as was uſed in ſubtraction of integers, applies here, 10 in any place being equal to 1 of the next ſuperior claſs.

MULTIPLICATION OF DECIMALS.

159. Rule. *Multiply the numbers together, (whether they be pure decimals, or integers with decimals annexed)*

annexed) as if there were no decimal point; and point off as many places of the product for decimals, as there are decimal places in both factors together. If there be not ſo many places in the product, complete the number by prefixing cyphers to the left.

Ex. 1ſt. Multiply 1.37 and 1.8 together.

$$\begin{array}{lr} & 1.37 \\ & 1.8 \\ \hline & 1096 \\ & 137 \\ \hline \text{Product} & 2.466 \\ \hline \end{array}$$

For $1.37 = \frac{137}{100}$ and $1.8 = \frac{18}{10}$.

Now, $\frac{137}{100} \times \frac{18}{10} = \frac{137 \times 18}{1000} = \frac{2466}{1000}$, which, by decimal notation, is 2.466.

Ex. 2d. Multiply .016 and 125 together;

$$\begin{array}{lr} & 125 \\ & .016 \\ \hline & 750 \\ & 125 \\ \hline \text{Product} & 2.000 \\ \hline \end{array}$$

For $.016 = \frac{16}{1000}$; and $\frac{16}{1000} \times 125 = \frac{2000}{1000}$ $= 2.000$ or 2.

In general, if in the multiplier there be n, and in the multiplicand there be m decimal places, the multiplier is equal to that integral number, which reſults from taking away the decimal point, divided by 1 with n cyphers; and the multiplicand is equal to that integral number, which reſults from taking away the decimal point, divided by 1 with m cyphers; therefore the product of the factors

$$= \frac{\text{the product of theſe integral numbers}}{\text{1 with } n \text{ cyphers} \times \text{1 with } m \text{ cyphers}}$$

$$= \frac{\text{Product of the integral numbers}}{\text{1 with } \overline{m+n} \text{ cyphers}}\text{; which is,}$$

(by decimal notation) the product of the integral numbers, having $\overline{m+n}$ places pointed off for decimals.

160. Problem. *To reduce a vulgar fraction to a decimal.*

Rule. *Add cyphers at pleaſure to the right hand of the numerator, divide by the denominator, and cut off from the quotient as many places for decimals, as cyphers were added to the numerator.*

Ex. 1ſt. $\frac{3}{4} = \frac{300}{4} \div 100 = 75 \div 100 = .75.$

Ex. 2d. $\frac{12}{5} = \frac{120}{5} \div 10 = 24 \div 10 = 2.4.$

If there be not ſo many places in the quotient, complete the number by prefixing cyphers to the left.

Ex. 3d. $\frac{1}{25} = \frac{100}{25} \div 100 = 4 \div 100 = .04.$

General demonſtration. If n cyphers be annexed to the numerator, the value of the fraction is multiplied by 1 with n cyphers. This increaſed value, when obtained by actual diviſion, muſt therefore be divided by 1 with n cyphers; i. e. (by decimal notation) n places muſt be pointed off for decimals.

Obſervation. Whatſoever number of cyphers be added, the value of the reſult will be the ſame.

Thus, in the 1ſt example, if the numerator had been multiplied by 1000, inſtead of 100, the anſwer would have come out .750, which is equal to .75. But the moſt convenient method for practice is, not to add cyphers, ſo long as the diviſion can be carried on without them, and afterwards only as the operation may require them. All the figures of the quotient, which can be obtained without the addition of cyphers, will be integers; the decimal point being then put down, the next ſignificant figure muſt be the 1ſt, 2d, or lower place of decimals, according as it may be neceſſary to add one, two, or more cyphers to the remainder, in order to obtain it; and in like manner all the ſucceeding figures of the quotient are to be determined, as if each cypher annexed was taken down from a dividend. And when there is no remainder, the operation ceaſes.

Ex. 1ſt. Reduce $\frac{161}{16}$ to a decimal.

$$\begin{array}{l} 16\ )\ 161\ (\ 10.0625 \\ \quad\ \ \underline{16} \\ \qquad\ \ 100 \\ \qquad\ \ \ \underline{96} \\ \qquad\quad\ 40 \\ \qquad\quad\ \underline{32} \\ \qquad\qquad 80 \\ \qquad\qquad \underline{80} \\ \qquad\qquad \cdot\ \cdot \end{array}$$

Ex. 2d. Reduce $\frac{4}{625}$ to a decimal.

$$\begin{array}{l} 625\ )\ 4000\ (\ .0064 \\ \qquad\quad \underline{3750} \\ \qquad\quad\ 2500 \\ \qquad\quad\ \underline{2500} \\ \qquad\qquad \cdot\ \cdot \end{array}$$

Ex. 3d. Reduce $\frac{22}{3}$ to a decimal.

3) 22 (7·333 &c.

Ex. 4th. Reduce $\frac{3}{7}$ to a decimal.

7) 30 (.428571428571 4 &c,

Ex. 25th

Ex, 5th. Reduce $\frac{1}{11}$ to a decimal.

11) 1.00 (.090909 &c.

161. In many caſes, as in the three laſt examples, the vulgar fraction cannot be reduced to a fraction with a decimal denominator; or, the value of the fraction cannot be exactly made up of integers, 10$^{\text{ths}}$, 100$^{\text{ths}}$, &c. but the decimal part will go on as far as we pleaſe to continue the operation; the ſame figure or figures recurring, and in the ſame order; and it is evident, that as ſoon as a dividual ariſes which has occurred before, the repetition will begin. Now ſuch dividual will neceſſarily ariſe, unleſs the operation terminate; becauſe the remainders never being ſo great as the diviſor, the whole number of remainders poſſible, cannot amount to a number equal to the diviſor.

Theſe interminate decimals, with recurring figures, are called recurring or circulating decimals.

DIVISION OF DECIMALS.

162. Rule. *Conſider the diviſor and dividend as whole numbers forming a vulgar fraction, of which find the decimal quotient by art.* 160; *remove the decimal point of this quotient as many places to the left, as there were decimal places in the dividend more than in the diviſor, or as many places to the right*

right as there were decimal places in the diviſor more than in the dividend, i. e. in the former caſe divide, and in the latter caſe multiply this quotient by unity with a number of cyphers annexed, equal to the difference of the number of decimal places in the diviſor, and the number of decimal places in the dividend.

Ex. 1ſt. Divide 320.04 by 6.4.

$\frac{32004}{64}$ reduced according to art. 160, is 500.0625; ∴ the quotient required is 50.00625.

For, ſince 32004 ÷ 64 = 500.0625, 32004 ÷ 64 = 5000.625, (where the decimal point is removed one place to the right) becauſe 6.4 is but a 10th part of 64. And becauſe 320.04 is but an 100th part of 32004, 320.04 ÷ 6.4 will be but an 100th part of 32004 ÷ 6.4, i. e. will be equal to an 100th part of 5000.625, or equal to 50.00625 (the decimal point being removed two places to the left); and the decimal point is at laſt found removed from the place determined by the reduction of the fraction $\frac{32004}{64}$, as many places to the left as there are decimal places in 320.04 more than in 6.4.

Ex. 2d. Divide 36 by .012.

$\frac{36}{12} = 3$, or 3.000 &c.; therefore 36 diivded by .012, which is but a 1000th part of 12, is 3000.0 &c. or 3000; the decimal point being removed three places to the right.

General demonſtration. If the dividend have *m* decimal places, and the diviſor have *n*, the dividend is equal to that *integral* number, which reſults from taking away the decimal point, divided by 1 with *m* cyphers; and the diviſor is equal to *its integral* number divided by 1 with *n* cyphers; therefore $\frac{\textit{dividend}}{\textit{diviſor}} =$

$$\frac{\overline{\textit{dividend integral}} \div \overline{1 \textit{ with } m \textit{ cyphers}}}{\overline{\textit{diviſor integral}} \div \overline{1 \textit{ with } n \textit{ cyphers}}} =$$

$\frac{\textit{divided integral}}{\textit{diviſor integral}} \times \frac{1 \textit{ with } n \textit{ cyphers}}{1 \textit{ with } m \textit{ cyphers}}$, which, if *m* be greater than *n*, is equal to $\frac{\textit{dividend integral}}{\textit{diviſor integral}}$, $\overline{\div 1 \textit{ with } m-n \textit{ cyphers}}$; but if *n* be greater than *m*, is equal to $\frac{\textit{dividend integral}}{\textit{diviſor integral}} \overline{\times 1 \textit{ with } \overline{n-m} \textit{ cyphers}}$.

i. e. the decimal point muſt be removed from the place, determined by the reduction of the fraction, $\frac{\textit{dividend integral}}{\textit{diviſor integral}}$, $\overline{m \sim n}$ places to the left or right, in the reſpective caſes.

---

* The claſs of each figure of a quotient may be determined by conſidering in what claſſes the diviſor and dividend ſtand, i. e. what claſſes their loweſt figures ſtand in. If they ſtand in the ſame claſs, the quotient figure muſt be in the claſs of units. If the dividual ſtand one, or two, or more claſſes higher than the diviſor, the quotient figure muſt be the claſs of tens; in the claſs of hundreds; &c. in the reſpective caſes. If the dividual ſtand one, or two, or more claſſes

CHAPTER V. PART III.

OF DIVISION.

ARTICLE 163. AS quotients cannot always be expreſſed in integral terms, it was impoſſible to exhibit an accurate and complete view of the nature of diviſion, without introducing the principles of fractions; on which account it became neceſſary to confine the obſerva-

---

claſſes lower than the diviſor, the quotient figure muſt be in the claſs of tenths; in the claſs of hundreds; &c. in the reſpective caſes.

For example, Firſt, Let it be required to divide 3.027 by 2.5. Here the dividual, 3.0, or 30 tenths, and the diviſor, 25 tenths, both ſtand in the ſame claſs; therefore the value of the quotient figure muſt be the ſame as the value of the quotient figure of 30 divided by 25, or in the claſs of units.

Secondly, Let it be required to divide 3.027 by .25. Here the dividual ſtands in the claſs of tenths, and the diviſor in the claſs of hundredths; therefore the value of the quotient figure muſt be ten times the value of the quotient figure of 30 divided by 25, or in the claſs of tens.

Thirdly, Let it be required to divide 4 or 4.000 &c. by 57. Here the dividual, 4.00 ſtands in the claſs of hundredths, and the diviſor ſtands in the claſs of units; therefore the value of the quotient figure muſt be an hundredth part of the value of the quotient figure of 400 divided by 57, or in the claſs of hundredths.

This conſideration forms an eaſy rule for decimal diviſion and reduction, for when the claſs of the firſt figure of the quotient is determined, the claſſes of the reſt follow in courſe.

tions and examples in the firſt part of this chapter to ſuch quantities, as gave integral quotients. It is proper therefore now to reſume the ſubject, and illuſtrate it with various examples, firſt obſerving, that *a quotient is that quantity, which, multiplied into the diviſor (according to the moſt extenſive ſenſe of multiplication) produces the dividend.*

164. In all caſes whatſoever of diviſion, the operation may be denoted by a ſymbol. And quantities with ſuch a ſymbol attached to them, repreſent the *reſult* of that operation.

165. If the dividend be under the form of a multiple of the diviſor, the operation may actually take place, as was explained in the firſt part of this chapter.

166. If the dividend be not under ſuch form, a ſymbol muſt be employed: but ſtill the quotient may be exhibited under different forms, inaſmuch as fractions are reducible to others of equal value in different terms, and reſolvable into parts.

167. If the diviſor and dividend have a common factor, it may be expunged from each, and the expreſſion be thus reduced to more ſimple terms.

Thus, $\frac{ac}{ab}=\frac{c}{b}$; $\frac{5a}{5b}=\frac{a}{b}$; $\frac{10a^2}{5ab}=\frac{2a}{b}$;

$\frac{ax^2-ay^2}{ac}=\frac{x_2-y^2}{c}$.

168. If the dividend be a compound quantity, each term may be ſeparately divided by the diviſor.

$$\text{Thus, } \frac{x^2-y^2}{c} = \frac{x^2}{c} - \frac{y^2}{c}.$$

And it may happen, that ſome of theſe reſulting fractions, ſhall be expreſſible in integral terms, though all be not.

$$\text{Thus, } \frac{cx+by}{x} = \frac{cx}{x} + \frac{by}{x} = c + \frac{by}{x}.$$

169. If the diviſor be compound, it may happen that part of the quotient ſhall be expreſſible in integral terms and part of it not; or there may be no part of it expreſſible in integral terms. In this laſt caſe it will generally be moſt convenient to ſet down the diviſor and dividend as a fraction, having firſt divided them by any common factor, in order to ſimplify the expreſſion. But if it be thought proper, the quotient, even in this caſe, may be expreſſed by a ſeries of fractions: for there muſt be ſome fraction, the product of whoſe multiplication into the diviſor, ſhall contain, among its terms, ſome one term of the dividend: again, there muſt be ſome fraction the product of whoſe multiplication into the diviſor ſhall contain ſome one of the terms of the remainder; and ſo on. And this operation may be continued on as far as

is thought neceſſary; and $\frac{\textit{the laſt remainder}}{\textit{diviſor}}$ added to the quotient obtained, will always make up the whole quotient of the given quantities.

In the ſame manner, when the quotient can partly be expreſſed in integral terms, it may be continued on by the fractional terms, as far as may be thought neceſſary.

170. In order to obtain quotients from compound diviſors in the ſimpleſt terms, and moſt regular manner, *the diviſor and dividend muſt be arranged according to the powers of ſome letter common to both; and ſuch a letter ſhould be choſen as has higher powers in the dividend than in the diviſor: the quotient of the firſt term of the diviſor, and the firſt term of the dividend, muſt be ſet down as the firſt term of quotient; the product of this term multiplied into the diviſor muſt be ſubtracted from the dividend; and the remainder (arranged according to the powers of the ſame letter) muſt be proceeded with in the ſame manner; and ſo on, till there be no remainder, or till it appear that there always will be one; in which caſe the operation may be continued as far as is thought expedient, and* $\frac{\text{the laſt remainder}}{\text{diviſor}}$ *muſt be added to the quotient obtained.*

EXAMPLES.

EXAMPLES.

1ſt. Divide $a^2-2ab+b^2$ by $a-b$

$$
\begin{array}{rl}
a-b\ )^{*} & a^2-2ab+b^2\ (a-b \\
 & a^2-ab \\
\cline{2-2}
 & \quad -ab+b^2 \\
 & \quad -ab+b^2 \\
\cline{2-2}
 & \qquad \cdot\ \ \cdot \\
\cline{2-2}
\end{array}
$$

Ex. 2. Divide $6a^3-48$ by $2a^2+4a+8$

$$
\begin{array}{rl}
2a^2+4a+8\ ) & 6a^3-48\ (\ 3a-6 \\
 & 6a^3+12a^2+24a \\
\cline{2-2}
 & -12a^2-24a-48 \\
 & -12a^2-24a-48 \\
\cline{2-2}
 & \qquad \cdot\ \ \cdot \\
\cline{2-2}
\end{array}
$$

---

[*] If the terms of the dividend be diſpoſed in this order, $-2ab+a^2+b^2$, the quotient may be obtained, but in terms leſs ſimple.

$$
\begin{array}{rl}
a-b\ ) & -2ab+a^2+b^2\ (\ -2b+a+b \\
 & -2ab\quad +2b^2 \\
\cline{2-2}
 & \quad a^2-b^2 \\
 & \quad a^2-ab \\
\cline{2-2}
 & \quad ab-b^2 \\
 & \quad ab\quad b^2 \\
\cline{2-2}
 & \qquad \cdot\ \ \cdot \\
\cline{2-2}
\end{array}
$$

Ex. 3d. Divide a^2+x^2 by $a+x$

$$
\begin{array}{l}
a+x\ )\ a^2+x^2\ (\ a-x+\dfrac{2x^2}{a}-\dfrac{2x^3}{a^2} \\
\qquad\qquad a^2+ax \\
\hline
\qquad\qquad -ax+x^2 \\
\qquad\qquad -ax-x^2 \\
\hline
\qquad\qquad\qquad +2x^2 \\
\qquad\qquad\qquad +2x^2+\dfrac{2x^3}{a} \\
\hline
\qquad\qquad\qquad\qquad -\dfrac{2x^3}{a} \\
\qquad\qquad\qquad\qquad -\dfrac{2x^3}{a}-\dfrac{2x^4}{a^2} \\
\hline
\text{Remainder} \qquad\qquad\qquad\quad +\dfrac{2x^4}{a^4} \\
\hline
\end{array}
$$

By obſerving the manner, in which the terms ariſe, their law may be diſcovered; and the quotient may be continued on *ad libitum*, without farther diviſion: In this example, the ſigns will be + and − alternately, and the terms, after the firſt two, will be $2x$, multiplied into the ſucceſſive powers of $\frac{x}{a}$.

If a be greater than x, the farther this ſeries be extended, the leſs will it differ in value from the true quotient; for after *n* terms, the remainder will be $\frac{2x^{n-1}}{a^{n-2}}$; and this quantity, divided by the diviſor, completes the quotient: Now, if $\frac{x}{a}$ be a proper fraction, the greater *n* is the leſs is *$\frac{x^{n-1}}{a^{n-1}}$; conſequently the leſs is $\frac{x^{n-1}}{a^{n-2}}$; and $\frac{x^{n-1}}{a^{n-2}} \div \overline{a+x}$; i. e. the leſs does the ſum of the preceding terms differ from the exact quotient. On the contrary, if *x* be greater than *a*, the farther the diviſion is carried, the more will the ſeries differ from the true quotient; for then the greater *n* is, the greater is $\frac{x^{n-1}}{a^{n-2}} \div \overline{a+x}$, and to have the ſeries in this caſe converge in value to the true quotient, the terms of the dividend and diviſor muſt be arranged according to the powers of *x*; for that arrangement will produce the ſeries $x-a+\frac{2a^2}{x}-$ &c. where, (upon ſuppoſition that *x* is greater than *a*) the remainders decreaſe.

---

* For if $\frac{x}{a}$ be a proper fraction, or a part of unity, $\frac{x}{a} \times \frac{x}{a}, = \frac{x^2}{a^2}$, is a part of $\frac{x}{a}$; $\frac{x^2}{a^2} \times \frac{x}{a}, = \frac{x^3}{a^3}$, is a part of $\frac{x^2}{a^2}$, &c.

Ex. 5th. Divide 1 by $1-x$.

$$
\begin{array}{lll}
1-x\ ) & 1\ (& 1+x+x^2 \\
 & \underline{1-x} & \\
 & \quad x & \\
 & \quad \underline{x-x^2} & \\
 & \qquad x^2 & \\
 & \qquad \underline{x^2-x^3} & \\
\text{Rem.} & \qquad \underline{\ \ x^3\ \ } &
\end{array}
$$

In this example the terms of the quotient appear under an integral form, the firſt term of the diviſor being unity. If x be leſs than 1, each remainder will be leſs than the preceding, and the farther the diviſion be carried on, the more nearly will the ſeries exhibit the true quotient.

Suppoſe $x = \frac{1}{2}$, then $1-x = \frac{1}{2}$; and $\frac{1}{1-x} = 1 \div \frac{1}{2} = 2$.

The ſeries will be $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. and after any number of terms, the ſum of all that could follow, if the diviſion were carried on indefinitely, is not greater than the laſt found term; for ſince the remainders are always poſitive, the ſum of the ſeries is always leſs than the true quotient by the quantity $\frac{\textit{remainder}}{\textit{diviſor}}$, i. e. the ſum of all the terms, that would follow is not greater than $\frac{\textit{remainder}}{\textit{diviſor}}$; Now, after n terms, the remainder is $x^n = \frac{1}{2^n}$; therefore $\frac{\textit{remainder}}{\textit{diviſor}} = \frac{1}{2^n} \div \frac{1}{2} = \frac{1}{2^{n-1}}$ = the laſt term.

If the arrangement of the diviſor had been $-x+1$, the quotient ſeries would have been $-\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} -$ &c. which, if x be leſs than 1, diverges from the true quotient, but converges, upon ſuppoſition that x is greater than 1. The whole quotient would be $-\frac{1}{x} - \frac{1}{x^2} -$ &c.

$-\frac{1}{x^n} + \frac{1}{x^n} \div \overline{1-x}.$

Ex. 6th.

$$\begin{array}{rl} y-1\ ) & y^3-1\ (\ y^2+y+1 \\ & \underline{y^3-y^2} \\ & \quad y^2-1 \\ & \quad \underline{y^2-y} \\ & \qquad y-1 \\ & \qquad \underline{y-1} \\ & \qquad \underline{\ \cdot\ \cdot\ } \end{array}$$

If the ſame power of the leading letter be found in two or more terms, the operation may ſometimes be carried on more ſimply and conveniently by collecting ſuch terms into one. (See art. 85.)

Ex. 7th. Divide $a^3-ca^2+cba-2b^2a+b^3$ by $a-b$

$$\begin{array}{lll}
a-b\,) & a^3-ca^2+\overline{cb-2b^2}.\,a+b^3 & (a^2+\overline{b-c}.\,a-b^2,\text{or} \\
 & a^3-ba^2 & \quad a^2+ba-ca-b^2 \\ \hline
 & \overline{b-c}.\,a^2 & \\
 & \overline{b-c}.\,a^2+\overline{cb-b^2}.\,a & \\ \hline
 & \qquad\qquad -b^2a & \\
 & \qquad\qquad -b^2a+b^3 & \\ \hline
 & \qquad\qquad \cdot \quad \cdot & \\ \hline
\end{array}$$

Ex. 8th.

$$\begin{array}{lll}
x-a\,) & x^3-px^2+qx-r & (\,x^2+\overline{a-p}.\,x+q-ap+a^2 \\
 & x^3-ax^2 & \\ \hline
 & \overline{a-p}.\,x^2 & \\
 & \overline{a-p}.\,x^2+\overline{ap-a^2}.\,x & \\ \hline
 & \qquad\qquad \overline{q-ap+a^2}.\,x & \\
 & \qquad\qquad \overline{q-ap+a^2}.\,x-qa+pa^2-a^3 & \\ \hline
 & \text{Rem.}\qquad\qquad a^3-pa^2+qa-r & \\ \hline
\end{array}$$

CHAPTER VI.

OF INVOLUTION AND EVOLUTION.

ARTICLE 164. IF a quantity be multiplied into itſelf, it is ſaid to be involved or raiſed. The products are called *powers*, and the quantity by whoſe multiplication into itſelf theſe powers are generated, is called the *root*.

165. For the ſake of brevity, powers are expreſſed by placing above the root, a figure denoting the number of factors. This figure is called the *Index* or *Exponent*, and from it the power is denominated: Thus,

a	is called the	1ſt.	power of the root a, and is otherwiſe expreſſed by	a^1
$a\ a$		2d.		a^2
$a\ a\ a$		3d.		a^3
$a\ a\ a\ a$		4th.		a^4
&c.		&c.		&c.
a a a &c. to n factors		n^{th}		a^n

The 2d and 3d powers are generally called the *ſquare* and *cube* reſpectively, and the 4th, 5th, and 6th powers are ſometimes called the *biquadrate*, *ſurſolid*, and *cubocube* reſpectively.

166. *Any*

46

166. *Any quantity is raiſed to a propoſed power by raiſing each factor to that power.*

For as often as the given quantity is multiplied into itſelf, ſo often will each factor in it be repeated.

EXAMPLES.

The ſquare of ab $(ab \times ab)$ is $a^2 b^2$;

The cube of ab is $a^3 b^3$;

The n^{th} power of ab ($= ab \times ab \times$ &c. to n factors $= aa$ &c. to n factors $\times\, bb$ &c. to n factors) $= a^n b^n$.

The ſquare of $2ab$ is $4a^2b^2$; its cube is $8a^3 b^3$.

$\overline{3ab^2}\rceil^2 = 9a^2b^4$. $\overline{2a^2b^3c^4}\rceil^5 = 2^5a^{10}b^{15}c^{20}$.

The ſquare of $\frac{2a}{b}$ $\left(\frac{2a}{b} \times \frac{2a}{b}\right)$ is $\frac{4a^2}{b^2}$; its cube is

$\frac{8a^3}{b^3}$. $\overline{\frac{b^2mx^2}{c^3}}\rceil^3 = \frac{b^6m^3x^6}{c^9}$

$\overline{\overline{a^2+x^2} \times \overline{a^2-x^2}}\rceil^2 = \overline{a^2+x^2}\rceil^2 \times \overline{a^2-x^2}\rceil^2$.

Cor. Where the quantity to be involved is a fraction, both numerator and denominator are raiſed to the propoſed power.

167. *Powers of the ſame root are multiplied together, by adding together their exponents for a new exponent.*

For there will be as many factors in the product, as there are factors in the quantities to be multiplied together.

Thus, $a^2 \times a^3$ $(= aa \times aaa = aaaaa) = a^5$.

$a \times -a^2$, or $a^1 \times -a^2$ $(= a \times -aa = -aaa) = -a^3$.

$a^n \times a^m$ $(= aa \ldots\ldots$ &c. to n factors $\times\, aa \ldots$ &c. to m factors $= aa \ldots$, &c. to $\overline{m+n}$ factors$) = a^{m+n}$

168. Converſely, *Any power is divided by a leſs power of the ſame root, by ſubtracting the index of the diviſor from the index of the dividend, for a new index.*

Thus, $\frac{a^5}{a^2}\left(= \frac{aaaaa}{aa} = aaa\right) = a^3 = a^{5-2}$.

$$\frac{a^{m+n}}{a^m}\left(= \frac{aa \text{ \&c. to } \overline{m+n} \text{ factors}}{aa \text{ \&c. to } m \text{ factors}} = aa \text{ \&c. to } n \text{ factors}\right) = a^n = a^{m+n-m}.$$

169. If the rule in the laſt article be made *general*, whether the diviſor be or be not a *leſs* power than the dividend, the reciprocal of any power will be expreſſed by prefixing the negative ſign to its index.

For the greater index of the diviſor ſubtracted from the leſs index of the dividend, will leave their difference with a negative ſign; and a leſs power divided by a greater, is equal to unity divided by that power, whoſe index is the difference of the indices of diviſor and dividend.

Thus, (if the rule be made general) $\frac{a^2}{a^4}$, $= \frac{1}{a^2}$ will be denoted by (a^{2-4} or) a^{-2}; $\frac{a^n}{a^m}$, which, if n be greater than m, is equal to a^{n-m}, but equal to $\frac{1}{a^{m-n}}$ if m be greater than n, will, in both caſes, be denoted by a^{n-m}, where the index, $n-m$, will be poſitive, on the former ſuppoſition, but negative on the latter.

170. If the index of the diviſor be equal to the index of the dividend, the index of the quotient derived from this rule will be 0; for $\frac{a^1}{a^1}$ will be $a^{1-1} = a^0$; $\frac{a^n}{a^n}$ will be $a^{n-n} = a^0$.

$\frac{a^1}{a^1}$ or $\frac{a^n}{a^n}$ is equal to 1; therefore a^0 will ſignify 1. And the ſeries

aaa.... to n factors, aa... to $n-1$ factors, &c., a, 1, $\frac{1}{a}$, $\frac{1}{a^2}$, &c, where each term is equal to the preceding divided by a, will be expreſſed by a ſeries, the indices of whoſe terms decreaſe by unity, viz. by the ſeries

a^n, a^{n-1}, &c. a^2, a^1, a^0, a^{-1}, a^{-2}, &c.

171. Theſe powers of $\frac{1}{a}$, or reciprocals of the powers of a, when they are expreſſed by negative exponents, are called negative powers of a. It is obvious that they are multiplied and divided by the ſame rules as poſitive powers, ſince from the application of thoſe rules are their exponents derived.

Thus, $a^{-m} \times a^{-n}$, or $\frac{1}{a^n} \times \frac{1}{a^m} = \frac{1}{a^{m+n}} = a^{-m-n}$, where the index, $-m-n$, is the ſum of $-m$ and $-n$.

$a^{-m} \times a^n$, or $\frac{1}{a^m} \times a^n = \frac{a^n}{a^m} = a^{n-m}$.

Alſo, $a^{-m} \div a^{-n}$, or $\frac{1}{a^m} \div \frac{1}{a^n} = \frac{a^n}{a^m} = a^{n-m}$, where the index, $n-m$, is $-n$ ſubtracted from $-m$.

$a^{-m} \div a^n$ or $\frac{1}{a^m} \div a^n = \frac{1}{a^{m+n}} = a^{-m-n}$, where the index, $-m-n$, is n ſubtracted from $-n$.

$a^m \div a^{-n}$, or $a^m \div \frac{1}{a^n} = a^{m+n}$, where the index, $m+n$, is $-n$ ſubtracted from m.

172. As to the ſigns of powers it is obvious that *When the quantity to be involved is* poſitive, *all its powers will be* poſitive. *And when the quantity to be involved is* negative, *all its powers whoſe exponents are* odd numbers, *will be* negative. *But all its powers, whoſe exponents are* even numbers *will be* poſitive.

For the product of *two* negative factors is *poſi-tive*; therefore the product of *four* (= the product of *two* × the product of *two*) is poſitive; therefore the product of *ſix* (= the product of *four* × the product of *two*) is poſitive; and, in general, the product of *any* even number of negative factors is poſitive.

The powers of $-a$, and of $-\frac{1}{a}$ are $-a+a^2-a^3+a^4-a^5+$ &c. $-a^{-1}+a^{-2}-a^{-3}+$ &c.

173. From the laſt and 167th articles it appears, that

Any quantity is raiſed to any propoſed power by multiplying the index of each factor by the exponent of the power, and making the reſult poſitive, *except where the given quantity is negative, and the exponent of the propoſed power at the ſame time an* odd number.

For to raiſe any quantity to the 2d, 3d, 4th, &c. power is (by art. 167.) to add the exponents of its factors once, twice, thrice, &c. to themſelves; therefore the ſecond power of any quantity is had by doubling each exponent, the 3d power by tripling each exponent, and *in general*, the m^{th} power is had by multiplying each exponent by m.

Thus, the ſquare of a or a^1, is $a^{1\times 2}$ or a^2; its cube is $a^{1\times 3}$ or a^3; its m^{th} power is $a^{1\times m}$, or a^m.

a^m, raiſed to the n^{th} power, is a^{mn}.

$2a^{-3}$, raiſed to the n^{th} power, is $2^n a^{-3n}$.

$a^m b^n c$, raiſed to the power r, is $a^{mr} b^{nr} c^r$.

$-a^m$, raiſed to the n^{th} power, (or $-a^m \times -a^m \times$ &c. to n factors) is $\pm a^{mn}$, where the negative ſign is to be prefixed, if n be an odd number.

$\dfrac{3a^2b}{4c}$, raiſed to the m^{th} power, is $\dfrac{3^m a^{2m} b^m}{4^m c^m}$.

174. If the quantity to be involved be compound, the involution may be repreſented by the index, or the quantity may be actually multiplied into itſelf, by the rule in art. 84.

Thus, the 2d power of $a+b$ (or $\overline{a+b} \times \overline{a+b}$) is $\overline{a+b}^{\,2}$, or $a^2+2ab+b^2$.

Powers of the *Binomial* $a+b$

$$\begin{array}{l}
a+b \quad \text{the root.} \\
a+b \\
\hline
a^2+ab \\
\quad +ab+b^2 \\
\hline
a^2+2ab+b^2 \quad \text{the ſquare, or ſecond power.} \\
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad +a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 \quad \text{the cube, or third power.} \\
a+b \\
\hline
\end{array}$$

$a^4+3a^3b+3a^2b^2+ab^3$
$\quad +a^3b+3a^2b^2+3ab^3+b^4$

$a^4+4a^3b+6a^2b^2+4ab^3+b^4$ the fourth power.
$a+b$

$a^5+4a^4b+6a^3b^2+4a^6b^3+ab^4$
$\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ the 5th power
$a+b$

$a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5$
$\quad +a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6$

$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$
&c. the 6th power.

175. If the powers of $a-b$ be required, they will be found the ſame as the preceding, only the terms in which the exponent of b is an odd number, will be found negative, becauſe an odd number of negative factors produces a negative product.

Thus, the cube of $a-b$ will be found to be $a^3-3a^2b+3ab^2-b^3$. Where the 2d and 4th terms are negative, the exponent of b being an odd number in theſe terms. In general, *The terms of any power of* a − b *are poſitive and negative alternately*.

176. The

176. The reverſe of involution, or the reſolving of powers into their roots, is called *Evolution*. The ſquare root of any quantity, a, is that quantity, which multiplied into itſelf produces a; The cube root of any quantity, a, is that quantity, whoſe cube, or 3d power is a; in general, *The nth root of any quantity*, a, *is that quantity, whoſe nth power is* a.

177. Roots are expreſſed by placing before the quantity, whoſe root is required, the ſymbol $\surd$ (called *The Radical Sign*) and prefixing at the head a figure, to indicate the denomination of the root. Thus,

The ſquare root of a is expreſſed by $\sqrt[2]{a}$ or $\sqrt{a}$.

The cube root of a, by $\sqrt[3]{a}$.

The 4th root of a^2b-x^2, by $\sqrt[4]{a^2b-x^2}$.

The 5th root of 2 by $\sqrt[5]{2}$.

In general, the nth root of a, by $\sqrt[n]{a}$.

178. The above is the general notation for roots in all caſes: but if the quantity, whoſe ſquare, cube, or higher root is required, be the ſquare, cube, or higher power of ſome other quantity, that other quantity (it is evident) is the root required; and the radical ſign need not be employed. Thus,

The ſquare root of 4 ($\sqrt{4}$) is 2 or -2;

The ſquare root of a^2 ($\sqrt{a^2}$) is a or $-a$;

The ſquare root of a^{2n} is $\pm a_{n}$;
The ſquare root of a^{-2} is $\pm a^{-1}$;
The ſquare root of $\frac{9a^2x^6}{b^4}$ is $\frac{3ax^3}{b^2}$;
The cube root of $8\ (\sqrt[3]{8})$ is 2;
The cube root of $-8\ (\sqrt[3]{8})$ is -2;
The cube root of $\frac{64a^{3n}b^6}{27c^{3m}}$ is $\frac{4a^nb^2}{3c^m}$;
The 4th root of $a^{4n}\ (\sqrt[4]{4^{4n}})$ is $\pm a^n$;
The nth root of $\frac{a^n}{b^n}\ (\sqrt[n]{\frac{a^n}{b^n}})$ is $\frac{a}{b}$;
The nth root of $\overline{a^2-x^2}^{2n} \times \overline{a^2+x^2}^{3n}$ is $\overline{a^2-x^2}^{2} \times \overline{a^2+x^2}^{3}$

In general, converſely to art. 166, any propoſed root is had, by extracting that root of each factor in the given quantity. Hence, (converſely to art. 173,) the following

Rule. *Divide the index of each factor in the given quantity by the number that denominates the root required.*

179. The ambiguity in the ſign of any root, that is denominated by an even number, is the neceſſary conſequence of admitting abſtract negative quantities. For by the definitions of multiplication $-a \times -a = +a \times +a = a^2$; therefore, by the definition of roots, both $-a$ and $+a$ are ſquare roots of a^2. Alſo, $-a \times -a \times -a \times -a = +a \times +a \times +a \times +a = a^4$; therefore both $-a$ and $+a$ are 4th roots of a^4. &c. *In general*,

Thoſe roots of a poſitive quantity, which are denominated by an even number, may be either poſitive or negative.

180. If the number that denominates the root required be *odd*, the ſign of the root muſt be the *ſame* as the ſign of the given quantity. For an odd number of factors, all affected with the ſame ſign, muſt be *poſitive*, in order to produce a *poſitive product*, and *negative* in order to produce a *negative product*.

Thus, $\sqrt[3]{8}$ is 2; $\sqrt[3]{-8}$ is -2.

181. If the number that denominates the root required be *even*, and the given quantity *negative*, the root is *impoſſible*. Becauſe a *negative* product cannot reſult from the multiplication of an *even* number of factors, which are all affected with the ſame ſign.

182. Theſe *impoſſible roots* are employed in Algebra, and are denoted by the *radical ſign* prefixed to the negative quantity.

Thus, $\sqrt{-a^2}$ is called the ſquare root of $-a^2$.

183. The root of a product is equal to the product of the roots of the factors, and *vice verſa*.

Thus, $\sqrt[n]{abc} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c}$.

184. The root of a quotient is equal to the root of the dividend divided by the root of the diviſor.

Thus, $\sqrt[n]{\frac{ab}{c}} = \frac{\sqrt[n]{ab}}{\sqrt[n]{c}}$.

185. The n^{th} root of the m^{th} power of any quantity, a, is equal to the m^{th} power of the n^{th} root of that quantity.

For, the n^{th} root of a^m is equal to $\sqrt[n]{a} \times \sqrt[n]{a} \times$ &c. to m factors; but $\sqrt[n]{a} \times \sqrt[n]{a} \times$ &c. to m factors, is, by the definition of powers, the m^{th} power of $\sqrt[n]{a}$; $\therefore \sqrt[n]{a^m} = \overline{\sqrt[n]{a}}\rceil^{m}$.

186. The m^{th} root of the n^{th} root of any quantity, a, is equal to the mn^{th} root of a.

For, let $a = x^{mn}$, then $\sqrt[n]{a} = \sqrt[n]{\overline{x^{mn}}} = x^m$; and the m^{th} root of $\sqrt[n]{a} =$ the m^{th} root of $x^m = x = mn^{th}$ root of (x^{mn} or of) a.

187. The bn^{th} root of the bm^{th} power of a is equal to the n^{th} root of the m^{th} power of a.

For the bn^{th} root of $a^{bm} =$ (by art. 186.) the n^{th} root of the b^{th} root of $a^{bm} =$ the n^{th} root of a^m.

188. If $\frac{m}{n} = \frac{s}{r}$, $\sqrt[n]{a^m} = \sqrt[r]{a^s}$.

For if $\frac{m}{n} = \frac{s}{r}$, multiplying each by nr, $\frac{mnr}{n} = \frac{snr}{r}$, or $mr = sn$; but, (by art. 187) $\sqrt[n]{a^m} = \sqrt[nr]{a^{mr}}$ and $\sqrt[r]{a^s} = \sqrt[rn]{a^{sn}} = \sqrt[nr]{a^{mr}}$; $\therefore \sqrt[n]{a^m} = \sqrt[r]{a^s}$.

189. If the rule in art. 178, be made general, whether the quotients ariſing from the diviſion of the indices, be integral or not, a new notation, by fractional indices, for expreſſing roots of quanti-

ties will ariſe, which will be attended with great convenience in reſpect to the management of indices, as it will extend to roots and powers of roots, the rules for multiplying and dividing powers of the ſame quantity.

For the ſquare root, cube root, 4th root, &c. .... nth root of any quantity a, or a^1, will be deſignated by $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, &c. ....$a^{\frac{1}{n}}$.

The ſame roots a^{-1}, will be, $a^{-\frac{1}{2}}$, $a^{-\frac{1}{3}}$, &c. .... $a^{-\frac{1}{n}}$.

The ſame roots of $a^{\pm m}$ will be $a^{\pm\frac{m}{2}}$, $a^{\pm\frac{m}{3}}$, $a^{\pm\frac{m}{4}}$, &c. $a^{\pm\frac{m}{n}}$. And inſtead of theſe fractions, any other fractions whatſoever, that are equal to them, may be ſubſtituted. (See art. 188.)

Let any two roots then whatſoever of the ſame quantity, a, be repreſented $a^{\frac{m}{n}}$ and $a^{\frac{c}{d}}$ where m, n, c, and d may be poſitive or negative. I ſay that $a^{\frac{m}{n}} \times a^{\frac{c}{d}}$ is equal to a, with an index equal to $\frac{m}{n} + \frac{c}{d}$; i. e. $a^{\frac{m}{n}} \times a^{\frac{c}{d}} = a^{\frac{md+nc}{nd}} =$ the ndth root of

* $a^{md * nc}$. Alſo, that $a^{\frac{m}{n}} \div a^{\frac{c}{d}}$ is equal to a with an

* + index equal to $\frac{m}{n} - \frac{c}{d}$; i. e. $a^{\frac{m}{n}} \div a^{\frac{c}{d}} = a^{\frac{md-nc}{nd}}$.

For $a^{\frac{m}{n}} \times a^{\frac{c}{d}}$ (by art. 187.) $= a^{\frac{md}{nd}} \times a^{\frac{nc}{nd}}$ (the nd^{th} root of $a^{md} \times$ the nd^{th} root of a^{nc}) = the nd^{th} root of $a^{md} \times a^{nc}$ = the nd^{th} root of a^{md+nc}, which, according to this fractional notation, is $a^{\frac{md+nc}{nd}}$.

Alſo, $a^{\frac{m}{n}} \div a^{\frac{c}{d}} = a^{\frac{md}{nd}} \div a^{\frac{nc}{nd}}$ = the nd^{th} root of $a^{md} \div a^{nc} = \sqrt[nd]{a^{md-nc}} = a^{\frac{md-nc}{nd}}$.

190. Roots, when thus deſignated by fractional exponents, are called fractional powers, alſo ſometimes imperfect powers.

Thus, the ſquare root of a, when deſignated by $a^{\frac{1}{2}}$, is called a to the power $\frac{1}{2}$; and $a^{\frac{m}{n}}$ is called a to the power $\frac{m}{n}$.

191. Theſe fractional powers (it is obvious) are involved and evolved by the ſame rules as integral powers.

Thus, $a^{\frac{m}{n}}$ raiſed to the c^{th} power $= a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times$ &c. to c factors $= a^{\frac{m}{n} \times c}$ And the c^{th} root of $a^{\frac{m}{n}}$ is equal to $a^{\frac{m}{n} \div c} = a^{\frac{m}{nc}}$, becauſe $a^{\frac{m}{nc}}$, raiſed to the c^{th} power is $a^{\frac{m}{nc} \times c} = a^{\frac{m}{n}}$.

192. From the deſignation of the *reciprocals* of powers by negative indices, it follows that

Any factor may be transferred from the numerator *of a fraction to the denominator, and* vice verſa, *if the ſign of its index be changed.*

For if any factor be expunged from the numerator of a fraction, and its reciprocal be made a factor of the denominator, or any factor be expunged from the denominator of a fraction, and its reciprocal be made a factor of the numerator, the value of the fraction will not be changed.

Thus, $\frac{ac}{bd} = \frac{a}{bd \times \frac{1}{c}} = \frac{ac \times \frac{1}{b}}{d}$.

EXAMPLES.

$$a \left(\text{or } \frac{1 \times a^1}{1} = \frac{1}{1 \times a^{-1}}\right) = \frac{1}{a^{-1}};$$

$$\frac{ax}{b} = \frac{a}{bx^{-1}} = axb^{-1} = \frac{1}{a^{-1}x^{-1}b};$$

$$\frac{a^2x^5}{b^3}\left(= \frac{a^2 \times x^2 \times x^3}{b^3}\right) = \frac{a^2x^2}{b^3x^{-3}} = \frac{a^2}{x^{-5}b^3} = \frac{a^2x^5b^{-2}}{b}$$

$$= a^2x^5b^{-3} = \text{\&c.}$$

$$\frac{a^n}{b^m} = \frac{a^{n-r}}{b^m \times a^{-r}} = \frac{a^{n+r}}{b^m a^r} = \frac{a^{n-n}}{b^m a^{-n}};$$

$$\frac{a^3}{b\sqrt{a^2-x^2}}\left(\text{or } \frac{a^3}{b \times \overline{a^2-x^2}\rceil^{\frac{1}{2}}}\right) = a^3 \times \overline{a^2-x^2}\rceil^{-\frac{1}{2}} \times b^{-1}.$$

193. If the quantity, whoſe root is required, be compound, the root may be expreſſed by the radical ſign, or by a fractional index, or it may be actually extracted. If the given compound quantity be ſuch as would reſult from the involution of ſome other compound quantity to a power of the ſame denomination as the propoſed root, that other quantity is evidently the root required. And if there be no ſuch exact root, yet, in all caſes, a ſeries may be found, which, if its terms decreaſe in value, ſhall approximate to the root required.

194. The operation of Evolution being the reverſing of the ſteps in the operation of Involution, the method of finding the terms of the root is to be diſcovered by attending to the involution of compound quantities.

OF THE EXTRACTION OF THE SQUARE ROOTS OF COMPOUND QUANTITIES.

195. The ſquare of $a+b$ is $a^2+2ab+b^2=a^2+\overline{2a+b}\times b$; the ſquare of $a-b$ is $a^2-2ab+b^2=a^2+\overline{2a-b}\times -b$; therefore the ſquare of $a+b+c$, or of $\overline{a+b}+c$, conſidering $a+b$ as one term, is $\overline{a+b}^2+2.\overline{\overline{a+b}+c}\times c$, and the ſquare of $a+b+c+d$, or of $\overline{a+b+c}+d$, is $\overline{a+b+c}^2+2.\overline{\overline{a+b+c}+d}\times d$;

and ſo on for any number of terms; a new member introduced into the root increaſing the ſquare by $\overline{\textit{twice the previous members}} \times \overline{\textit{the new member}}$ $+ \overline{\textit{new member}}\rceil^{2}$

If therefore the ſquare of $a \pm b \pm c \pm$ &c. be diſtinguiſhed off after the ſquare of a, after the ſquare of $a \pm b$, after the ſquare of $a \pm b \pm c$, and ſo on, it will ſtand thus,

$$a^2,\ + \overline{2\,a \pm b} \times \pm b,\ + \overline{2.\overline{a \pm b} \pm c} \times \pm .c,\ + \text{\&c.}$$

where the firſt term is the ſquare of the firſt member of the root; and if the remaining quantities were worked out at length, the firſt term after each comma would be $2\,a \times \overline{\textit{a member of the root}}$; therefore the other members of the root are to be obtained by dividing theſe firſt terms by $2a$. Hence the following,

196. Rule for extracting the ſquare root of a compound quantity.

*Arrange the terms of the given quantity orderly.** *Find the ſquare root* (a) *of the firſt term, which ſet down for the firſt member of the root. Subtract its ſquare* (a^2) *from the given quantity, and divide the firſt term of the remainder by its double*, (2a); *the*

---

* The arrangement is as follows: Thoſe terms where the firſt or leading letter is found alone, muſt ſtand firſt; thoſe where only the firſt or ſecond letter is found, before thoſe where the other letters, (if there be more) are found; and ſo on: And the order of the leading letter is from the higheſt to the loweſt power, but of the others from the loweſt to the higheſt power.

quotient (b) *is the ſecond member of the root. Add this ſecond member to the double of the firſt, multiply their ſum* (2a+b) *by the ſecond member* (b), *and ſubtract the product* ($2ab+b^2$) *from the laſt remainder. If nothing remain, the root is obtained, but if there be a remainder, divide the firſt term of it, by double the firſt member of the root* (2a); *the quotient* (c) *is the third member of the root. Add this third member to the ſum of double of the firſt and ſecond, and multiply their ſum,* (2a+2b+c) *by the third member* (c), *and proceed as before, always dividing the firſt term of the remainder by* (2a), *the double of the firſt member for a new member, and ſubtracting* $\overline{\textit{double the previous part}+\textit{new member}}\times\overline{\textit{new member}}$ *from the laſt remainder, for a new remainder.*

Note, The algebraical letters in the above rule, which are included in parentheſes, are not a neceſſary part of it. They exhibit the rule briefly and perſpicuouſly upon the ſuppoſition that *a*,*b*,&c. are the members of the root. Other letters, which occur in different examples, may be accommodated to this notation by ſubſtitution. Thus, if the firſt term of the given quantity be x^4, x^4 correſponds with a^2, and x^2 with *a*.

EXAMPLES.

1st.

$$\begin{array}{rl}
 & a^2-2ab+b^2\ (a-b \\
 & a^2 \\ \hline
2a-b\ ) & -2ab+b^2 \\
 & -2ab+b^2 \\ \hline
 & \cdot\ \cdot \\ \hline
\end{array}$$

2d.

$$\begin{array}{rl}
 & a^2+2ab+b^2-2ac-2bc+c^2\ (\,a+b-c \\
 & a^2 \\ \hline
2a+b\ ) & 2ab+b^2-2ac-2bc+c^2 \\
 & +2ab+b^2 \\ \hline
2a+2b-c\ ) & -2ac-2bc+c^2 \\
 & -2ac-2bc+c^2 \\ \hline
 & \cdot\ \cdot\ \cdot \\ \hline
\end{array}$$

3d.

$$\begin{array}{rl}
 & a^2-ax+\frac{x^2}{4}\ (a-\frac{x}{2} \\
 & a^2 \\ \hline
2a-\frac{x}{2}) & -ax+\frac{x^2}{4} \\
 & -ax+\frac{x^2}{4} \\ \hline
 & \cdot\ \cdot \\ \hline
\end{array}$$

4th.
$$\begin{array}{ll}
 & a^4+2a^3+a^2-2a^2b-2ab+b^2\;(a^2+a-b \\
 & a^4 \\ \hline
2a^2+a\,) & 2a^3+a^2-2a^2b-2ab+b^2 \\
 & +2a^3+a^2 \\ \hline
2a^2+2a-b\,) & -2a^2b-2ab+b^2 \\
 & -2a^2b-2ab+b^2 \\ \hline
 & \cdot\qquad\cdot \\ \hline
\end{array}$$

5th.
$$\begin{array}{ll}
 & 9x^2+\frac{3ax}{2}+\frac{a^2}{16}\;(3x+\frac{a}{4} \\
 & 9x^2 \\ \hline
6x+\frac{a}{4}\,) & \frac{3ax}{2}+\frac{a^2}{16} \\
 & \frac{6ax}{4}+\frac{a^2}{16} \\ \hline
 & \cdot\qquad\cdot \\ \hline
\end{array}$$

6th. Let the root of $1+x$ be required.

It is evident that no expreſſion whatſoever, ſimple or compound, can be accurately equal to $\sqrt{1+x}$ i. e. no expreſſion whatſoever, multiplied into itſelf, can produce $1+x$; therefore an approximation only can be made to the root.

According to the rule, the operation will be as follows.

$1+x$ 7

$$\begin{array}{l}
1+x\ \big)\ 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\&c. \\
\quad 1 \\
\hline
2+\frac{x}{2}\big)\ x \\
\qquad x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8}\big)\ -\frac{x^2}{4} \\
\qquad\qquad -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2+x-\frac{x^2}{4}+\frac{x^3}{16}\big)\ +\frac{x^3}{8}-\frac{x^4}{64} \\
\qquad\qquad +\frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{\overline{16}|^2} \\
\hline
\qquad\qquad -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{\overline{16}|^2} \\
\hline
\end{array}$$

From the work it appears, that if the given quantity, inſtead of being $1+x$, had been $1+x+\frac{x^2}{4}$ after the ſecond ſubtraction there would have been no remainder, and the root would have been $1+\frac{x}{2}$; and that if it had been $1+x * -\frac{x^3}{8}+\frac{x^4}{64}$, after the third ſubtraction there would have been no remainder, and the root would have been $1+\frac{x}{2}-\frac{x^2}{8}$; and that if it had been $1+x\ *\ *+\frac{5x^4}{64}-\frac{x^5}{64}+\frac{a^6}{16}\rceil^2$, after the fourth ſubtraction there would have been no remainder, and the root would have been $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$; and ſo on. If therefore x be of ſuch a magnitude, that of theſe ſucceſſive quantities, $1+x+\frac{x^2}{4}$, $1+x\ *-\frac{x^3}{8}+\frac{x^4}{64}$, $1+x\ *\ *+\frac{5x^4}{64}-\frac{x^5}{64}-\frac{x^6}{16^2}\rceil$ &c., each ſucceeding one differs leſs from $1+x$, than the preceding, the ſeries $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-$ &c. approximates in value to the true root of $1+x$, or, the more terms of the ſeries be taken, the leſs will their ſum differ from the ſquare root of $1+x$.

7th. If the ſquare root of $a+b$ be required, the ſeries found according to the rule, will be $a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}}-$&c. where the exponents of a in the denominators, and of b in the numerators, increaſe by unity.

$a+b=a\times\overline{1+\frac{b}{a}}\ \therefore\sqrt{\overline{a+b}}=\sqrt{a}\times\sqrt{1+\frac{b}{a}}$; and the ſame ſeries as the above will reſult, if the

root

root of $1+\frac{b}{a}$, extracted in the ſame manner as the root of $1+x$ in example 6th, be multiplied into $\sqrt{a}$.

For by that example it appears that

$$\overline{1+\frac{b}{a}}\rceil^{\frac{1}{2}} = 1+\frac{b}{2a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\&c.$$

$$\therefore a^{\frac{1}{2}}\times\overline{1+\frac{b}{a}}\rceil^{\frac{1}{2}} \text{ is } a^{\frac{1}{2}}\times\overline{1+\frac{b}{2a}-\frac{b^2}{8a^2}+\&c,}$$

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$$= a^{\frac{1}{2}}+\frac{ba^{\frac{1}{2}}}{2a}-\frac{b^2a^{\frac{1}{2}}}{8a^2}+\frac{b^3a^{\frac{1}{2}}}{16a^3}-\&c,$$

$$= a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}}-\&c,$$

OF THE CUBE ROOT OF COMPOUND QUANTITIES.

197. The cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$; and the cube of $a-b$ is $a^3-3a^2b+3ab^2-b^3$; therefore the cube of $a\pm b\pm c$, or of $\overline{a\pm b}\pm c$ (conſidering $a\pm b$ as one term) is $\overline{a\pm b}\rceil^{3} + 3.\overline{a\pm b}\rceil^{2}\times\pm c + 3\times\overline{a\pm b}\times c^2 \pm c^3$ and ſo on for any number of terms in the root, a new member (m) introduced into the root, increaſing the cube by $3\times\overline{\text{prior terms}}\rceil^{2}\times m+3\times\overline{\text{prior terms}}\times m^2+m^3$. If then the cube

of $a\pm b\pm c\pm$ &c. be diſtinguiſhed off after the cube of a, after the cube of $a\pm b$, after the cube of $a\pm b\pm c$, and ſo on, it will ſtand thus:

$$a^3,\ \pm 3a^2b+3ab^2\pm b^3,\ +3.\overline{a\pm b}|^2\times\pm c+3.\overline{a\pm b}\times c^2$$
$$\pm c^3,\ +3.\overline{a\pm b\pm c}|^2\times\pm d+3.\overline{a\pm b\pm c}\times d^2\pm d^3,\ \&c.$$

where the firſt term is the cube of the firſt member of the root, and if the remaining quantities were worked out at length, the firſt term after each comma would be $3a^2\times\overline{\textit{a member of a root}}$.

Therefore the other members of the root are to be obtained by dividing theſe firſt terms by $3a^2$; hence the following

198. Rule for extracting the cube root of a compound quantity.

Having arranged the given quantity, the cube root (a) *of the firſt term* (a^3) *is the firſt member of the root required; ſubtract its cube* (a^3) *from the given quantity, and divide the firſt term* ($3a^2b$) *of the remainder by* ($3a^2$) *triple its ſquare; the quotient* (b) *is the ſecond member of the root. Subtract three times* $\overline{\textit{the firſt member}}|^2\times\overline{\textit{ſecond member}}$ + *three times* $\overline{\textit{the firſt member}}\times\overline{\textit{ſecond member}}|^2+\overline{\textit{ſecond member}}|^3$ ($3a^2b+3ab^2+b^3$) *from the laſt remainder, and if there be another remainder, divide the firſt term of it by* ($3a^2$) *triple* $\overline{\textit{the firſt member}}|^2$*; the quotient* (c) *is the third member of the root. Subtract from the* *laſt remainder three times* $\overline{\textit{firſt + ſecond member}}^2 \times \overline{\textit{third member}}$ *+ three times* $\overline{\textit{firſt and ſecond member}} \times \overline{\textit{third member}}^2 + \overline{\textit{third member}}^3$, $(3.\overline{a+b}^2 \times c + 3.\overline{a+b} \times c^2 + c^3)$ *and proceed as before. Always finding the new member by dividing the firſt term of the remainder by* $(3a^2)$ *triple* $\overline{\textit{the firſt member}}^2$*, and ſubtracting three times* $\overline{\textit{the previous members}}^2 \times \overline{\textit{new member}}$ *+ three times the previous members* $\times \overline{\textit{new member}}^2$*, +* $\overline{\textit{new member}}^3$*, for a new remainder.*[*]

Ex. 1ſt. Extract the cube root of

$$8x^3 - 12x^2b + 6xb^2 - b^3$$

$$\begin{array}{rl} & 8x^3 - 12x^2b + 6xb^2 - 6^3 \quad (\; 2x - b \\ & 8x^3 \\ \hline 12x^2\;) & -12x^2b + 6xb^2 - b^3 \\ & -12x^2b + 6xb^2 - b^3 \\ \hline & \qquad \cdot \qquad \cdot \\ \hline \end{array}$$

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[*] The above rule in effect is the ſame as the following:

Find the firſt member as above, and the other members by always ſubtracting the cube of thoſe already found from the original given quantity, *and dividing the firſt term of the remainders by triple* $\overline{\textit{the firſt member}}^2$.

For when $a^3 + 3a^2b + 3ab^2 + b^3$ has been ſubtracted from the given quantity, the whole cube of $\overline{a+b}$ will have been ſubtracted, and when $3.\overline{a+b}\rceil^2 \times c + 3.\overline{a+b} \times c^2 + c^3$ has alſo been ſubtracted, the whole cube of $a+b+c$ will have been ſubtracted.

Ex. 2d. Extract the cube root of $1+x$

$$\begin{array}{l} 1+x\ (\ 1+\frac{x}{3}-\frac{x^2}{9}+\&c. \\ 1 \\ \hline 3\ )\ *\ \ x \\ \qquad x+\frac{x^2}{3}+\frac{x^3}{27} \\ 3\ )-\frac{x^2}{3}-\frac{x^3}{27} \\ \qquad -\frac{x^2}{3}-\frac{2x^3}{9}-\frac{x^4}{27}+\frac{x^4}{27}+\frac{x^5}{81}+\frac{x^6}{9^3} \end{array}$$

$1+\frac{x}{3}$ is the cube root of the quantity $1+x+\frac{x^2}{3}+\frac{x^3}{27}$; $1+\frac{x}{3}-\frac{x^2}{9}$, is the cube root of the quantity $1+x-\frac{5x^3}{27}+\frac{x^5}{81}-\frac{x^6}{9^3}$, and ſo on; and if each of theſe quantities differ leſs from $1+x$ than the preceding, the ſeries $1+\frac{x}{3}-\frac{x^2}{9}$, +&c.approximates to the cube root of $1+x$.

Ex. 3d. If the cube root of $a+b$ be required, the ſeries found according to the rule will be $a^{\frac{1}{3}}+\frac{b}{3a^{\frac{2}{3}}}-\frac{b^2}{9a^{\frac{5}{3}}}+\frac{5b^3}{81a^{\frac{8}{3}}}-$ &c.

$\overline{a+b}\rceil^{\frac{1}{3}}=a^{\frac{1}{3}}+\times\overline{1+\frac{b}{a}}\rceil^{\frac{1}{3}}$; and the ſame ſeries will reſult if the cube root of $1+\frac{b}{a}$, extracted according to Ex. 2d, be multiplied into $a^{\frac{1}{3}}$.

Ex. 4th. Required the cube root of a^3-x^3. According to the laſt example, the ſeries will be

$$\overline{a^3}|^{\frac{1}{3}} - \frac{x^3}{3\overline{a^3}|^{\frac{2}{3}}} - \frac{x^6}{9\overline{a^3}|^{\frac{5}{3}}} - \&c.$$

$$= a - \frac{x^3}{3a^2} - \frac{x^6}{9a^5} - \&c.$$

OF THE EXTRACTION OF THE ROOTS OF COMPOUND QUANTITIES IN GENERAL.

199. It readily appears from the actual multiplication of any binomial, $a \pm b$ into itſelf, that the firſt term of the n^{th} power is a^n, and that the ſecond term is $+na^{n-1} \times \pm b$.

Since then $\overline{a\pm b}|^{n} = a^n + na^{n-1} \times \pm b + \&c.$

$\overline{a\pm b\pm c}|^{n} = \overline{\overline{a\pm b}\pm c}|^{n} = \overline{a\pm b}|^{n} + n.\overline{a\pm b}|^{n-1} \times \pm c$ $+ \&c.$

$= a^n + na^{n-1} \times \pm b + \&c. \quad + n.\overline{a\pm b}|^{n-1} \times \pm c + \&c.$

and $\overline{a\pm b\pm c\pm d}|^{n} = \overline{a\pm b\pm c}|^{n} + n.\overline{a\pm b\pm c}|^{n-1} \times \pm d$ $+ \&c.$

and ſo on. If therefore the n^{th} power of $a\pm b\pm c\pm$ &c. be worked out at length, and diſtinguiſhed off after the n^{th} power of a, after the n^{th} power of $a\pm b$, after the n^{th} power of $a\pm b\pm c$, and ſo on, the firſt term will be the n^{th} power of the firſt member of the root, and the firſt term after each of the diſtinguiſhing commas will be $na^{n-1}\times\overline{\textit{a member of the root}}$; therefore the other members of the root are to be found by dividing theſe firſt terms by na^{n-1}. Hence the following

200. Rule for extracting the n^{th} root of a Compound Quantity.

Having arranged the quantity, find the n^{th} *root* (a) *of the firſt term* (a^n); *this ſhall be the firſt member of the root; ſubtract its* n^{th} *power* (a^n) *from the given quantity, and divide the firſt term of the remainder by* n *times its* $\overline{n-1}|^{th}$ *power* (na^{n-1}); *the quotient* (b) *ſhall be the ſecond member of the root. Subtract* the $\overline{\text{firſt}+\text{ſecond member}}^{n}$ *from the given quantity, and if there be any remainder, proceed as before. Always dividing the firſt term of the remainder by* n *times the* $\overline{n-1}|^{th}$ *power of the firſt member of the root, and ſubtracting from the original given quantity the* n^{th} *power of the root obtained, for a new remainder.*

Ex. 1ſt. Required the ſquare root of $a^2+2ab+b^2$

$$a^2+2ab+b^2\ (\ a+b$$

$$a^2$$

$$2a\ )\quad 2ab+b^2$$

$$\overline{a+b}^2 = a^2+2ab+b^2$$

• •

Ex. 2d. Required the 5th root of $32x^{10}-80x^9+80x^8-40x^7+10x^6-x^5$

$$32x^{10}-80x^9+80x^8-40x^7+10x^6-x^5(2x^2-x$$

$$32x^{10}$$

$$80x^8\ )\ -80x^9$$

$$\overline{2x^2-x}\,\Big|^5=32x^{10}-80x^9+80x^8-40x^7+10x^6-x^5$$

• •

OF THE EXTRACTION OF THE ROOTS OF NUMBERS.

201. One principal uſe of the above inquiry into the methods of extracting the roots of compound Algebraic quantities, is the facilitating the inveſtigation of rules for extracting the roots of numbers,

bers. If, in the multiplication of numbers, the products of the parts were kept ſeparate, powers of numbers would appear under the ſame form as powers of algebraic terms; and the above rules might be applied without any deviation.

For example, if $\overline{23}|^{2}$ i. e. $\overline{20+3}|^{2}$ were exhibited under this form, $\overline{20}|^{2} + 2\times 20\times 3+3^{2}$, or, $400+120+9$, its ſquare root would be found by the rule in art. 196; but when theſe numbers, 400, 120 and 9, are incorporated, their ſum is 529, where neither the 400, nor the 120 appears ſeparately. Nevertheleſs, the ſquare root in this, and in all other examples, can be determined by a rule nearly the ſame as that above quoted, as will appear from the following conſiderations.

202. **Lemma.** Let m repreſent any whole number whatſoever. Then $10^{m} =$ 1 with m cyphers to the right hand, and $a\times 10^{m} = a$ with m cyphers to the right hand, $= a$ in the $\overline{m+1}|_{th}$ claſs of figures.

203. The ſquare root of 100 is 10; of 10000, 100; of 1000000, 1000; *in general*, the ſquare root of 1 with $2n$ cyphers, is 1 with n cyphers; from which conſideration it follows, that the ſquare root of any number leſs than 100, (i. e. of any number of *one* or *two* places) muſt conſiſt of *one* figure; of any number between 99 and 10000 (i. e. of any number of *three* or *four* places) of *two*

figures; of any number between 9999 and 1000000 (i. e. of any number of *five* or *ſix* places) of *three* figures; of any number not leſs than 1 with $\overline{2n-2}$ cyphers, but leſs than 1 with 2 *n* cyphers (i. e. of any number of $\overline{2\,n-1}$, or 2 *n* places) of *n* figures. If then a point be made over every ſecond figure in any number, beginning with the units, the number will be divided into as many periods, as there are points; each period conſiſting of two figures, except perhaps the left hand period, which will conſiſt of one figure, or two, according as the number of places in the given number be odd or even; and the number of periods or points will ſhew the number of places in the ſquare root.

204. Let the given number, whoſe ſquare root is required, conſiſt of *n* periods, then its root will conſiſt of *n* places of figures. Let the digits of the root be *a*, *b*, *c*, &c. *a* being in the higheſt claſs; *b* in the next; &c. then the root is *a* in the n^{th} claſs + *b* in the $\overline{n-1}|^{th}$ claſs + *c* in the $\overline{n-2}|^{th}$ claſs + &c. $= a \times 10^{n-1} + b \times 10^{n-2} + c \times 10^{n-3} +$ &c. where it is to be obſerved, that *a* muſt be one of the nine ſignificant figures; but *b*, *c*, &c. may, ſome or all of them, be cyphers; ſince the root may have cyphers in ſome or all of its inferior places.

I ſay that *a* is the higheſt digit poſſible, ſuch that its ſquare ſhall not exceed the firſt or left hand period. For taking *a* ſuch, $a \times 10^{n-1}$ is not greater than the root, becauſe $\overline{a\times 10^{n-1}}\rceil^{2}$, $=a^{2}\times 10_{2}^{n-2}$ $=a^{2}$ with $\overline{n-1}$ periods of cyphers to the left hand, is not greater than the given number. But a digit $(a+1)$ whoſe ſquare is greater than the firſt period, would be too great for the n^{th} claſs, becauſe $\overline{\overline{a+1}\times 10^{n-1}}\rceil^{2}$ $=\overline{a+1}\rceil^{2}$ with $\overline{n-1}$ periods of cyphers, is greater than the given number. And a digit $(a-1)$ which is leſs than a, would be too ſmall for the n^{th} claſs of the root, becauſe whatever figures occupy the inferior claſſes of the root, they cannot make up a number ſo great as *one* of the claſs in which a ſtands; conſequently $\overline{a-1}\times 10^{n-1}+$*the other members of the root* cannot be ſo great as $a\times 10^{n-1}$, i. e. cannot be ſo great as a number which is not greater than the root.

Therefore a is rightly determined by taking it the higheſt digit poſſible, whoſe ſquare is not greater than the left hand period.

I ſay that b is the higheſt figure poſſible, ſuch that $\overline{a\times 10^{n-1}+b\times 10^{n-2}}\rceil^{2}$ ſhall not be greater than the given number; or (becauſe $\overline{a\times 10^{n-1}+b\times 10^{n-2}}\rceil^{2}$ $=\overline{a\times 10^{n-1}}\rceil^{2}$ $+\overline{2a\times 10^{n-1}+b\times 10^{n-2}}\times b\times 10^{n-2}$) ſuch that if $\overline{a\times 10^{n-1}}\rceil^{2}$ be ſubtracted from the given number, $\overline{2a\times 10^{n-1}+b\times 10^{n-2}}\times b\times 10^{n-2}$

ſhall not be greater than the remainder; or, (becauſe $\overline{a \times 10^{n-1} + b \times 10^{n-2}}\rceil^2 = \overline{\overline{10a+b} \times 10^{n-2}}\rceil^2 = \overline{10a+b}\rceil^2 \times 10^{2n-4} = \overline{10a+b}\rceil^2$ with $\overline{n-2}$ periods of cyphers) ſuch that $\overline{10a+b}\rceil^2$ $(=100a^2 + \overline{20a+b} \times b)$ ſhall not be greater than the two firſt periods to the left; or ſuch that if a^2 be ſubtracted from the firſt period, and the ſecond period be annexed to the remainder, $\overline{20a+b} \times b$ ſhall not be greater than the reſulting number.

For let b be ſuch; then, by hypotheſis, no digit greater than b may ſtand in the $\overline{n-1}\rceil_{\text{th}}$ claſs. Neither may a leſs digit; becauſe the remaining inferior members of the root cannot be ſo great as *one* of the claſs in which b ſtands; conſequently $a \times 10^{b-1} + \overline{b-1} \times 10^{n-2}$ + *the other members of the root* cannot be ſo great as $a \times 10^{n-1} + b \times 10^{n-2}$; i.e. cannot be ſo great as a number which, by hypotheſis, is not greater than the root.

Therefore b is rightly determined by taking it the higheſt figure poſſible, ſuch that if a^2 be ſubtracted from the firſt period, and the ſecond period be annexed to the remainder, $\overline{20a+b} \times b$ ſhall not be greater than the reſulting number.

In a ſimilar manner (if there be more than two places in the root) it may be proved that c is the higheſt figure poſſible, ſuch that $\overline{a\times10^{n-1}+b\times10^{n-2}+c\times10^{n-3}}\rceil^{2}=\overline{\overline{100a+10b+c}\times10^{n-3}}\rceil^{2}=\overline{100a+10b+c}\rceil^{2}\times10^{2n-6}=\overline{100a+10b+c}\rceil^{2}$ with $\overline{n-3}$ periods of cyphers to the right hand, ſhall not be greater than the given number; or, (which is the ſame thing) ſuch that $\overline{100a+10b+c}\rceil^{2}$ ſhall not be greater than the three firſt periods; or, (becauſe $\overline{100a+10b+c}\rceil^{2}$ $=10000a^{2}+\overline{200a+10b}\times10b+\overline{200a+20b+c}\times c$ $=10000a^{2}+\overline{20a+b}\times b\times100+\overline{200a+20b+c}\times c$ $=a^{2}$ with two periods of cyphers $+\overline{20a+b}\times b$ with one period of cyphers $+\overline{200a+20b+c}\times c$) c muſt be the higheſt figure poſſible, ſuch, that if a^{2} be ſubtracted from the firſt period, and $\overline{20a+b}\times b$ be ſubtracted from the remainder with the ſecond period annexed to it, $\overline{20aa+20b+c}\times c$ ſhall not be greater than the number which reſults from annexing the third period to the remainder left after this ſecond ſubtraction. &c. &c.

Obſervation. It appears from the above, that the value of a, the higheſt figure of the root, may be eaſily aſcertained by inſpection.

If $\overline{a\times10^{n-1}}\rceil^{2}$ be ſubtracted from the given number, the remainder will conſiſt of $2a\times10^{n-1}\times b\times10^{n-2}$ +&c. but on account of the inferiority in the value of each ſucceeding figure in the root, the other terms of the remainder are ſmall in compariſon with this firſt; ſo that $b\times 10^{n-2}$ is nearly the quotient of this remainder divided by $2a\times 10^{n-1}$; or (becauſe $2a\times 10^{n-1}\times b\times 10^{n-2} = 2a\times 10^{2n-3}\times b$) b is nearly the quotient of this remainder divided by $2a\times 10^{2n-3}$; or, cutting off $\overline{2n-3}$ places from the remainder, (which nearly divides it by 10^{2n-3}) b is ſtill * more nearly the quotient of the part left divided by $2a$. But if $\overline{2n-3}$ places be cut off from the remainder, the part left is the very ſame as would reſult from ſubtracting a^2 from the firſt period, and annexing to the remainder the firſt figure of the ſecond period,

The value of b being aſcertained, let $\overline{a\times 10^{n-1}+b\times 10^{n-2}}\rceil^{2}$ ($=\overline{10a+b}\rceil^{2}\times 10^{2n-4} = \overline{10a+b}\rceil^{2}$ with $\overline{n-2}$ periods of cyphers) be ſubtracted from the given number, and the remainder (if there be more than two places in the root) will conſiſt of $2\times\overline{a\times 10^{n-1}+b\times 10^{n-2}}\times c\times 10^{n-3}$ + &c. $=2\times\overline{10a+b}\times 10^{2n-5}\times c$ + &c. but the other terms being inconſiderable with reſpect to this firſt, $c\times 10^{n-3}$ is nearly the quotient of this remainder divided by $2\times\overline{a\times 10^{n-1}+b\times 10^{n-2}}$; and c is nearly the quotient of this remainder divided by $2\times\overline{10a+b}\times 10^{2n-5}$; or cutting off $\overline{2n-5}$ places from the remainder, which nearly divides it by 10^{2n-5}) c is ſtill more nearly the quotient of the

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* The cutting off $2n-3$ places from any number, reduces it more than diviſion by 10^{2n-3}, unleſs the places cut off be cyphers.

part left divided by $2 \times \overline{10a+b}$. But if $\overline{a \times 10^{n-1} + b \times 10^{n-2}}|^2$ be ſubtracted from the given number, and $\overline{2n-5}$ places be cut off from the remainder, the part left will be the very ſame that would have reſulted from ſubtracting a^2 from the firſt period, annexing the ſecond period to the remainder, ſubtracting $2 \times \overline{10a+b} \times b$ from the reſulting number, and annexing the firſt figure of the third period to the number left after this ſecond ſubtraction. And (if the root conſiſt of more than three figures) in a ſimilar manner may the values of the ſucceeding places be aſcertained.

From the above inveſtigation is inferred the following

205. Rule for extracting the ſquare root of a number.

Divide the given number into periods, by placing a point over the units, another over the hundreds, and ſo on.

Find the higheſt digit, (a) *whoſe ſquare ſhall not exceed the firſt period to the left hand ; and place it to the right hand of the given number after the manner of a quotient figure in diviſion ; and it will be the firſt figure of the root required.*

Subtract its ſquare (a^2) *from the firſt period, and to the remainder annex the ſecond period, in order to form a dividend.*

Place the double of the figure of the root already ſound ($2a$) *to the left hand of the dividend ſor a diviſor.*

Divide the dividend, excluſive of the place of units, by this diviſor, and the quotient, or the quotient ſomewhat diminiſhed (b) *is the ſecond figure of the root.*

Annex this ſecond figure to the right hand of the diviſor, and multiply the reſulting number ($2a \times 10 + b$) *by it, and ſubtract the product* ($\overline{2a\times10+b}\times b$) *from the dividend.*

Note. *The true value of the ſecond figure of the root is that, which makes the above product either equal to the dividend, or as nearly as poſſible equal to it without exceeding it.*

If there be more than two periods in the given number, to the laſt remainder annex the third period in order to form a new dividend, and take for a new diviſor the double of the figures already found in the root; and find the third figure of the root, and the third remainder in the ſame manner from this new diviſor and dividend as the ſecond figure and ſecond remainder were found from the firſt diviſor and dividend.

Proceed thus till all the periods have been annexed. Always finding the new figure(m) *of the root, by bringing down to the remainder the next period, and conſidering what figure muſt be annexed to the double of the*

the figures already in the root, ſo that the reſult, $(\overline{\text{root obtained}} \times 10 + m)$ *when multiplied into the new figure, ſhall give a product either equal to the dividend, or as nearly as poſſible equal to it without exceeding it; which figure will always be nearly the quotient of the remainder with one figure of the ſucceeding period annexed to it, divided by the double of the figures already in the root. And always finding the new remainder by ſubtracting the ſaid product from the dividend.*

Note. If in the progreſs of the operation it be found that a cypher muſt occupy that place in the root, which is then to be determined, in order to ſave unneceſſary trouble, after having annexed the cypher to the diviſor employed in finding it, take the reſulting number for the new diviſor, and bring down the next period to the dividend, in order to find the ſucceeding figure of the root.

The moſt expeditious method of doubling the root obtained, is by adding the laſt figure of it to the laſt diviſor.

The following Table exhibits the ſquares and cubes of the nine digits:

Roots	1	2	3	4	5	6	7	8	9
Squares	1	4	9	16	25	36	49	64	81
Cubes	1	8	27	64	125	216	343	512	729

EXAMPLES.

1ſt. Required the ſquare root of 97344.

. . .
97344 (312 the root.
9

61 | 73
1 | 61

622 | 1244
| 1244

. .

2d. Required the ſquare root of 3481118001.

. . . . .
3481118001 (59001
25

109 | 981
9 | 981

118001 | 118001
| 118001

. .

It is evident that when unity annexed to the diviſor forms a number greater than the dividend, the next figure of the root muſt be a cypher.

3d. Required the ſquare root of 38453.

```
 . . .
38453 ( 196
1
------
29 | 284
 9 | 261
-----------
386 | 2353
    | 2316
-----------
       37 remainder
-----------
```

When there is a remainder after the laſt ſubtraction, the root obtained is not the root of the given number, but of the given number diminiſhed by that remainder. Thus, in the laſt example, 196 is not the ſquare root of 38453, but of $38453 - 37$ or 38416. Some fraction added to 196 will make up a number, which ſhall be more nearly the ſquare root of 38453 than 196 is; that is, whoſe ſquare ſhall differ leſs from 38453 than 38416 (the ſquare of 196) does. Such a fraction is found from the following conſideration—that becauſe $38453 = \frac{38453 \times m^2}{m^2}$ $\therefore$ the ſquare root of $38453 = \frac{\sqrt{38453 \times m^2}}{m}$; and if m be taken of ſuch a magnitude, that the neareſt defective root of

$38453 \times m^{2}$ (found by the rule) ſhall be greater * than $196m$,——equal ſuppoſe to $169m+r$; then $\frac{\sqrt{38453 \times m^{2}}}{m}$ will be found $= \frac{196m+r}{m} = 196 + \frac{r}{m}$; i. e. in addition to the root 196, we obtain the fraction $\frac{r}{m}$.

If m be taken $=$ *unity* with n cyphers to the right hand, m^{2} will $=$ *unity* with $2n$ cyphers, and $\frac{\sqrt{38453 \times m^{2}}}{m}$ will be $\frac{\sqrt{3845300 \text{ \&c. to } 2n \textit{ cyphers}}}{10 \text{ \&c. to } n \textit{ cyphers}}$ $=$ (by decimal notation) $\sqrt{3845300 \text{ \&c. to } 2n \textit{ cyphers}}$ having n places pointed off for decimals.

Now the firſt three figures of the root of 3845300 &c. are found by the ſame proceſs as the root of 38453; and the remaining figures, by bringing down the periods of cyphers. In general

---

* This muſt neceſſarily happen if m be taken a large number; for it is only required that $\overline{\overline{196}\rceil^{2} + 37} \times m^{2}$ ſhould be at leaſt as great as $\overline{196m + 1}\rceil^{2} = \overline{196}\rceil^{2} \times m^{2} + 2 \times 196m + 1$, or (ſubtracting $\overline{196}\rceil^{2} \times m^{2}$ from each), that $37m^{2}$ ſhould be as great as $2 \times 196m + 1$, or (dividing each by m) that $37m$ ſhould be as great as $2 \times 196 + \frac{1}{m}$.

If after the extraction of the root (according to the rule) a remainder be found, continue the operation at pleaſure, by annexing to that and to the ſucceeding remainders periods of cyphers, and point off as decimals the figures obtained by theſe additional periods.

Ex. 1ſt. Required the ſquare root of 2.

2 (1.414 &c.
1

24	100
4	96
281	400
1	281
2824	11900
	11296
	604

Ex. 2d. Required the ſquare root of 401.

```
        . .
        401 ( 20.0249
         4
     ----------------
   4002 | 10000
      2 |  8004
     ----------------
  40044 | 199600
      4 | 160176
     ----------------
 400489 | 3942400
      9 | 3604401
     ----------------
           337999
     ----------------
```

Neither in theſe nor in any other example can the decimal fraction ever terminate, becauſe no digit multiplied into itſelf produces a cypher in the place of units.

The root of any decimal is equal to the root of the number conſidered without the decimal point, divided by the root of the denominator of the decimal part.

Thus, $\sqrt{1.23} = \frac{\sqrt{123}}{\sqrt{100}}$

In order therefore to find the ſquare root of a decimal,

206. *Firſt make the number of decimal places even, (if it be not ſo already) by annexing a cypher to the right hand, in order that the ſquare root of the denominator may be* 10, *or ſome power of* 10, *and then find the root of the number conſidered as* integral, *and make that figure the firſt place of decimal, which was obtained from the firſt decimal period.*

For example, the root of .02 = the root of $\frac{2}{100} = \frac{1.414 \&c.}{10} = .1414$ &c.

The root of .002 = the root of .0020 = the root of $\frac{20}{10000} = \frac{4.47 \&c.}{100} = .0447$ &c.

The root of 1.44 = the root of $\frac{144}{100} = \frac{12}{10} = 1.2$.

207. The reaſoning, from which are derived the methods of extracting the higher roots of numbers, being exactly ſimilar to that employed in the preceding articles for the inveſtigation of the ſquare root, may be briefly and generally ſtated as follows:

The cube root of 1000 is 10; the cube root of 1000000 is 100; the cube root of 100 &c. to $3n$ cyphers is 100 &c. to n cyphers. Hence it follows, that the cube root of any number that has not more than *three* places of figures, will conſiſt of but *one* figure; that the cube root of any number that has more than *three* places of figures, but not more than *ſix*, will conſiſt of *two* figures; that the cube root of any number that has more than $\overline{n-1}\times 3$ places of figures, but not more than $n\times 3$ will conſiſt of n figures.

In general, let m denote the index of the root required, then becauſe the m^{th} root of 1 *with m cyphers* ($= m^{\text{th}}$ root 10^{m}) is 10; and the m^{th} root of 1 with 2 m cyphers ($=m^{\text{th}}$ root of 10^{2m}) is 100; and (in general) the m^{th} root of 1 with mn cyphers ($= m^{\text{th}}$ root of $10^{mn} = 10^{n}$) is 1 with n cyphers, it follows, that if the given number, whoſe m^{th} root is required, do not conſiſt of more than m places of figures, its m^{th} root will conſiſt of but *one* figure; that if it conſiſt of more than m places, but not more than 2 m places, its m^{th} root will conſiſt of *two* figures; *in general*, that if it conſiſt of more than $\overline{n-1}\times m$ places, but not more than $n\times m$ places, its m^{th} root will conſiſt of n figures.

208. Hence it follows, that if the given number, whoſe m^{th} root is required, be divided into periods by placing a point over every m^{th} figure, beginning from the units, viz. one over the units figure, another over the $\overline{m+1}$th figure, and ſo on, as many periods as are found, of ſo many figures will the m^{th} root conſiſt.

Let there be n periods in the given number; and let the figures of the root be a, b, c &c. a being in the higheſt claſs; b in the next; &c. then the root is a in the n^{th} claſs + b in the $\overline{n-1}\rceil$ th claſs + &c. or $a\times 10^{n-1} + b\times 10^{n-2}$ + &c. And the chief point neceſſary to be proved in order to eſtabliſh a rule, is,

1ſt. That a muſt be the higheſt digit poſſible, whoſe m^{th} power does not exceed the firſt or left hand period; or (which amounts to the ſame thing) that a muſt be the higheſt digit poſſible, ſuch that $\overline{a\times 10^{n-1}}\rceil^{m}$ ſhall not exceed the given number.

2dly. That b muſt be the higheſt figure poſſible, ſuch that $\overline{a\times 10^{n-1}+b\times 10^{n-2}}\rceil^{m} = \overline{\overline{10a+b}\times 10^{n-2}}\rceil^{m}$ ſhall not exceed the given number; or, (which is the ſame thing) ſuch that $\overline{10a+b}\rceil^{m}$ ſhall not exceed the two firſt periods.

3dly. That c (if there be more than two periods in the given number) muſt be the higheſt figure poſſible, ſuch that $\overline{a\times 10^{n-1}+b\times 10^{n-2}+c\times 10^{n-3}}\rceil^{m}$ $= \overline{\overline{100a+10b+c}\times 10^{n-3}}\rceil^{m}$ ſhall not exceed the given number, or, ſuch that $\overline{100a+10b+c}\rceil^{m}$ ſhall not exceed the three firſt periods. And ſo on.

The method of demonſtration is ſimilar to that in art. 204.

For taking a of the above magnitude, by hypotheſis, no digit greater than a can occupy the n^{th}

claſs of the root. Neither can a leſs; becauſe whatever figures ſtand in the inferior places of the root, the ſum of their values, not being ſo great as *one* of the claſs wherein a ſtands, cannot compenſate the deficiency.

And a ſimilar reaſoning applies to the ſucceeding figures of the root.

The firſt figure therefore of the m^{th} root of any given number will be known immediately, if the m^{th} power of each of the nine digits be known: e. g. if the left hand period be equal to or greater than the m^{th} power of 3, but not ſo great as the m^{th} power of 4, the firſt figure of the root will be 3.

In finding the ſucceeding figures of the root, conjecture and trial muſt be in ſome degree employed. But they may be confined within very narrow limits; for if a^m be ſubtracted from the firſt period, and the firſt figure of the ſecond period be annexed to the remainder, the ſecond figure of the root will be nearly equal to the quotient of the reſulting number divided by ma^{m-1}.

And if $\overline{10a+b}\rceil^{m}$ ($\overline{a \text{ with } b \text{ to the right hand}}\rceil^{m}$) be ſubtracted from the two firſt periods, and the firſt figure of the third period be annexed to the remainder, the third figure of the root will be nearly equal to the quotient of the reſulting number divided by $m \times \overline{10a+b}\rceil^{m-1}$. &c. &c.

For becauſe the two principal terms of the m^{th} power of $a\times 10^{n-1}+b\times 10^{n-2}+$ &c. are $a^m\times 10^{mn-m}$ and $m\times a^{m-1}\times 10^{\overline{n-1}\times\overline{m-1}}\times b\times 10^{n-2}$, it follows, that if a^m be ſubtracted from the firſt period, and the other periods (which being $\overline{n-1}$ in number, conſiſt of $mn-m$ places of figures) be annexed to the remainder, the reſulting number $(m\times a^{m-1}\times 10^{\overline{m-1}\times\overline{n-1}}\times b\times 10^{n-2}+$ &c.) divided by $(m\times a^{m-1}\times 10^{\overline{m-1}\times\overline{n-1}}\times 10^{n-2}$, or by) $m\times a^{m-1}\times 10^{mn-m-1}$ will not much exceed the ſecond figure, b; or, cutting off from the ſaid reſulting number $\overline{mn-m-1}$ figures, i. e. all the figures of the annexed periods, except the firſt figure of the ſecond period, b will be nearly the reſult of the part left divided by ma^{m-1}, as was aſſerted; and in a ſimilar manner, becauſe $\overline{a\times 10^{n-1}+b\times 10^{n-2}+c\times 10^{n-3}}\rceil^{m}=$ $\overline{\overline{10a+b}\times 10^{n-2}}\rceil^{m}+m\times\overline{\overline{10a+b}\times 10^{n-2}}\rceil^{m-1}\times c\times 10^{n-3}$ + &c. may the other aſſertions be proved.

Hence the following

209. **Rule for extracting the m^{th} root of a number.**

Divide the given number into periods, by placing a point over every mth *figure, beginning from the units, viz. one over the units, another over the* $\overline{\text{m}+1}\rceil$th *figure, and ſo on.*

Find the higheſt digit (a) *whoſe* m^{th} *power is not greater than the firſt or left hand period, and ſet it down for the firſt figure of the root required.*

Subtract its m^th^ *power* (a^{m}) *from the firſt period, and to the remainder annex the firſt figure of the ſecond period to form a dividend.*

Divide this dividend by m *times the* $\overline{m-1}\rceil$th *power of the firſt figure of the root* (ma^{m-1}) *and the quotient, or the quotient ſomewhat diminiſhed* (b) *will be the ſecond figure of the root.*

Place the ſecond figure of the right hand of the firſt, and ſubtract the m^th^ *power of the reſult* $(\overline{10a+b}\rceil^{m})$ *from the two firſt periods.*

Note. *The true value of the ſecond figure is that, which makes the ſaid* m^th^ *power* $(\overline{10a+b}\rceil^{m})$ *either equal to the two firſt periods, or as nearly as poſſible equal without exceeding them.*

If there be more than two periods in the given number, to the remainder left after the ſecond ſubtraction annex the firſt figure of the third period, and divide the reſulting number by m *times the* $\overline{m-1}\rceil$ *th power of the figures already in the root*$(m\times\overline{10a+b}\rceil^{m-1})$ *in order to find the third figure,* (c); *which muſt be ſuch, that when placed to the right hand of the figures already obtained, the* m^th^ *power of the reſult* $(\overline{100a+10b+c}\rceil^{m})$ *ſhall be either equal to the three firſt periods, or as nearly as poſſible equal, without exceeding them.*

Proceed in the ſame manner till all the periods have been employed. Always annexing the firſt figure *of the ſucceeding period to the remainder, and dividing the reſult by* m *times the* $\overline{m-1}\rceil$ *th power of the figures already in the root, in order to find the next figure,* (n); *which muſt be ſuch, that being placed to the right hand of the figures already in the root, the* m^th^ *power of the reſulting number* $(\overline{\text{root obtained}\times 10+n}\rceil^{m})$ *ſhall be either equal to the periods that have been already employed, or as nearly as poſſible equal without exceeding them; and always ſubtracting the ſaid* m^th^ *power from the ſaid periods, in order to find the next remainder.*

*If there be a remainder at laſt, and the root be required to greater accuracy, continue the operation at pleaſure with periods of cyphers, and point off as decimals the figures obtained by theſe additional periods.**

---

* The reaſon is obvious; for let *R* repreſent any number, then

$$\sqrt[m]{R}=\sqrt[m]{\frac{R\text{ with }n\text{ periods of cyphers}}{100\text{ \&c. to }mn\text{ cyphers}}}=\frac{\sqrt[m]{R\text{ with }n\text{ periods of cyphers}}}{100\text{ \&c. to }n\text{ cyphers}}$$

$=\sqrt[m]{R\text{ with }n\text{ periods of cyphers}}$, having *n* places pointed off for decimals.

EXAMPLE.

EXAMPLES.

Required the cube root of 27054036008.

$$2\dot{7}05\dot{4}03\dot{6}00\dot{8} \ (\ 3002 \text{ the root.}$$
$$27$$

$$3\times 3^2 = 27 \ ) \ 0$$
$$\overline{30}|^3 = 27000$$

$$3\times\overline{30}|^2 = 2700 \ ) \ 540$$
$$\overline{300}|^3 = 27000000$$

$$3\times\overline{300}|^2 = 270000 \ ) \ 540360$$
$$\overline{3002}|^3 = 27054036008$$

---

210. The m^{th} root of a decimal = the m^{th} root of the number that reſults from taking away the decimal point, divided by the m^{th} root of the denominator of the decimal part.

Thus the 5th root of $1.05 = \dfrac{\sqrt[5]{105}}{\sqrt[5]{100}}$

In order therefore to obtain the m^{th} root of a decimal,

211. *Firſt, make the decimal period or periods complete, if they be not ſo already, by prefixing cyphers to the right hand, in order that the* mth *root of the denominator*

nator may be 10, *or ſome power of* 10, *extract the root of the number (ſo completed) conſidered as integral, by art.* 209, *&c. and make that figure the firſt place of decimals which was obtained by means of the firſt decimal period.*

Ex. 1ſt. Required the 4th root of 82.35721501.

$$
\begin{array}{rl}
 & 8\dot{2}.357\dot{2}150\dot{1}\ \ (\ 3.01 \\
 & 81 \\
\hline
4\times3^{3}=108\ ) & 13 \\
\overline{30}\rceil^{4}\quad & 810000 \\
\hline
4\times\overline{30}\rceil^{3}=108000\ ) & 135721 \\
\overline{301}\rceil^{4}\ = & 8235721501 \\
\hline
 & \cdot\quad\cdot \\
\hline
\end{array}
$$

Ex. 2. Required the cube of .0002=.000200.

The figures of the cube root of 200, found by art. 209, are 58 &c. ∴ the root required is .058 &c.

212. Becauſe one of the component parts of $\overline{10a+b}\rceil^{m}$ (ſo far as the ſignificant figures extend) is a^{m}, and (in the ſame manner) one of the component parts of $\overline{100a+10b+c}\rceil^{m}$ is $\overline{10a+b}\rceil^{m}$, and ſo on; therefore inſtead of the ſecond ſubtraction, as directed

in the rule, the *other* component parts of $\overline{10a+b}\rceil^{m}$ may be ſubtracted from the number that reſults from annexing the whole of the ſecond period to the firſt remainder, and inſtead of the third ſubtraction, as directed in the rule, the *other* component parts of $\overline{100a+10b+c}\rceil^{m}$ may be ſubtracted from the number that reſults from annexing the whole of the third period to the ſecond remainder; and ſo on. And in this it is that particular rules for the ſquare and cube roots differ from the foregoing general rule.

The component parts $\overline{10a+b}\rceil^{3}$ are $1000a^3$, $3\times100a^2b$, $3\times10ab^2$ and b^3; of $\overline{100a+10b+c}\rceil^{3}$ or $\overline{\overline{10a+b}\times10\times c}\rceil^{3}$ they are $\overline{10a+b}\rceil^{3}\times1000+3\times\overline{10a+b}\rceil^{2}\times c\times100$, $3\times\overline{10a+b}\times c^2\times10$, and c^3; and ſo on. * Hence is conſtructed the following,

---

* Particular rules might, on the ſame principle, be conſtructed for the higher roots; but the compotent parts of the higher powers of a binomial are too numerous to render ſuch rules ſerviceable.

It may be uſeful to obſerve, that if the index of the root required be the product of two or more numbers, inſtead of extracting the root at one operation, we may firſt extract that root whoſe index is one of the factors, and from the reſult extract that root whoſe index is another factor, &c.

Thus to obtain the 4th root of a number, we may extract the ſquare root of the ſquare root of it; to obtain the 6th root, we may extract the ſquare root of the cube root of it, or the cube root of the ſquare root of it; to obtain the eighth root we may extract the fourth root of the ſquare root of it, or the ſquare root of the 4th root of it, or the ſquare root of the ſquare root of the ſquare root of it.

213. Rule for extracting the cube root of a number.

Divide the given number into periods, by placing a point over every third figure beginning from the units.

Take for the firſt figure of the root the higheſt digit (a) *whoſe cube does not exceed the left hand period.*

Subtract its cube (a^3) *from that period, to the remainder annex the ſecond period, and call the reſulting number a reſolvend.*

Divide the reſolvend excluſive of the figures in the units and tens place by triple the ſquare of the figure in the root ($3a^2$) *and the quotient, or the quotient ſomewhat diminiſhed* (b) *will be the ſecond figure of the root.*

Multiply the triple ſquare of the firſt figure of the root into the ſecond figure, and place the product in the claſs of hundreds (or with two cyphers to the right hand.)

Multiply the triple of the firſt figure into the ſquare of the ſecond, and place the product in the claſs of tens, (or with one cypher to the right hand.)

Cube the ſecond figure of the root.

Subtract the ſum of the above three quantities ($3a^2b \times 100 + 3ab^2 \times 10 + b^3$) *from the reſolvend, and if there be more than two periods in the given* *number, to the remainder annex the third period to form a new reſolvend; and find the third figure of the root from the two already obtained, in the ſame manner as the ſecond figure was found by means of the firſt.*

Proceed thus till all the periods have been annexed, and continue the operation, if neceſſary, with periods of cyphers in order to obtain decimal places.

EXAMPLE.

Required the cube root of 206425072.

$20\dot{6}42\dot{5}07\dot{2}$ (591
$5^3 = 125$

81425 reſolvend
$3\times5^2 = 75$ diviſor
$3\times5^2\times9\times100 = 67500$
$3\times5\times9^2\times10 = 12150$
$9^3 = 729$

80379 ſubtrahend

10443 | 1046072 ſecond reſolvend
$3\times\overline{59}\rceil^2 =$ ~~10443~~ ſecond diviſor
$3\times\overline{59}\rceil^2\times1\times100 = 1044300$
$3\times59\times1^2\times10 = 1770$
$1^3 = 1$

1046071 ſecond ſubtrahend

1 remainder

CHAP.

CHAPTER VII.

OF EQUATIONS IN GENERAL, AND OF THE SOLUTION OF SIMPLE EQUATIONS.

214. A Propoſition aſſerting the equality of two algebraic quantities or expreſſions is called an *Equation.*

Thus $x-b=c$ and $x-a=0$ are equations; the former aſſerting the value of $x-b$ to be the ſame as the value of c, and the latter aſſerting the value of $x-a$ to be nothing.*

215. When a quantity ſtands alone on one ſide of an equation, the quantities on the other ſide are ſaid to be a *value* of it.

Thus, in the equation $x=b-d$, x ſtands alone on one ſide, and $b-d$ is a value of it.

---

* The aſſertion that any expreſſion is equal to 0, evidently implies that the ſum of the poſitive terms in that expreſſion is equal to the ſum of the negative terms, ſo as to counterbalance it. If $x+y=0$, *one* of the letters muſt denote a negative quantity, otherwiſe the propoſition would be abſurd. Suppoſe that by y, or $+y$, is meant the quantity -3, then the aſſertion is, that x with the addition of -3, is equal to 0.

216. If in any equation there be a quantity whoſe value is unknown, and by any management a new equation be deduced, one ſide of which is that unknown quantity, and the other ſide known quantities, that equation is ſaid to be *reſolved*, and the value of the unknown quantity is called a *root* of the equation.

217. Equations containing only one unknown quantity and its powers, are divided into *orders* according to the higheſt power of the unknown quantity to be found among its terms.

If the higheſt power of the unknown quantity to be found among any of the terms be the	1ſt.	The equation is called	*Simple, or of one dimention.*
	2d.		*Quadratic, or of two dim*^{ns}
	3d.		*Cubic, or of three dimentions,*
	&c.		*&c.*
	n^{th}		*Of* n *dimentions.*

218. Unknown quantities are uſually repreſented by the *laſt* letters of the alphabet z, y, x, &c. known quantities by the *firſt* a, b, c, &c.

219. The moſt common and uſeful application of algebra is in the inveſtigation of quantities that are unknown, from certain given relations to each other, and to ſuch as are known, and hence it has been called the *Analytical Art*. The equations employed for expreſſing theſe relations muſt therefore contain one or more unknown quantities, and the principal buſineſs of this art will be,

the deducing equations containing only one unknown quantity and reſolving them.

220. The following rules (which are true of *all* equations) are ſufficient for the ſolution of *ſimple* equations.

221. Rule 1ſt. *In any equation quantities may be tranſpoſed from one ſide to the other, if their ſigns be changed, and the two ſides will ſtill be equal.*

For if $x + a = b$, ſubtracting a from both ſides, $x + a - a = b - a$, or $x = b - a$, i. e. any *poſitive* quantity may be tranſpoſed, and placed on the other ſide with a *negative* ſign. Again, if $x - c = d$, adding c to both ſides, $x - c + c = d + c$, or $x = d + c$; i. e. any *negative* quantity may be tranſpoſed, and placed on the other ſide with a *poſitive* ſign.

222. Cor. If the ſigns of all the terms of an equation be changed, it will ſtill continue true; for this will have the ſame effect as tranſpoſing every term.

223. Rule 2d. *If every term on each ſide of an equation be multiplied by the ſame quantity, the ſides will ſtill be equal.*

Thus, if $a = b - c$, multiplying every term by n, $na = nb - nc$;

If $\frac{x}{a} = b + c$, multiplying every term by a, $x = \overline{b + c} \times a$.

For multiplying every term by any quantity, is multiplying the value of the whole ſide by that quantity.

224. Cor. An equation may be cleared of fractions by multiplying every term by the denominators of the fractions ſucceſſively,

Ex. Let $\frac{x}{a} + \frac{x}{b} = c$; multiply both ſides by a, and $x + \frac{ax}{b} = ac$; then multiply by b, and $bx + ax = abc$;

An equation may be cleared of fractions at one operation, by multiplying by the product of all the denominators, or by any quantity which is a common multiple of them all,

Ex. Let $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = m$, then multiplying by $a \times b \times c$ $\quad \frac{abcx}{a} + \frac{abcx}{b} + \frac{abcx}{c} = mabc$, or $bcx + acx + abx = mabc$.

Ex. 2. Let $\frac{x}{a} + \frac{x}{2a} + \frac{x}{4a} = r$; multiply by $4a$ and $4x + 2x + x = 4ar$.

Ex. 3. Let $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$; multiply by 12, and $6x + 4x + 3x = 12 \times 13 = 156$.

Rule

225. Rule 3d.* *If each ſide of an equation be divided by the ſame quantity, the ſides will ſtill be equal.*

Ex. 1ſt. Let $13x = 26$; divide both the ſides by 13, and $x = \frac{26}{13} = 2$.

Ex. 2d. Let $ax - x = b$, or $\overline{a-1}.\ x = b$; divide by $\overline{a-1}$, and $x = \frac{b}{a-1}$.

Ex. 3d. Let $x^2 - x = nx$ divide by x and $x - 1 = n$.

Cor. to the above three Rules.

226. If any quantity be found on both ſides of an equation with the ſame ſign, it may be taken away from each. Alſo, if every term on both ſides have a common multiplier or diviſor, that common multiplier or diviſor may be expunged.

Thus, if $ax + b = c + b$, $\quad ax = c$.

If $\frac{2ax^2}{b} + \frac{4acx}{3b} = \frac{2ad^2 - 2a^2d}{b}$, $\quad x^2 + \frac{2cx}{3} = d^2 - ad$

227. Rule 4. *If each ſide of an equation be raiſed to the ſame power, the ſides will ſtill be equal.*

Ex. 1. Let $x = 3$; then $x^2 = 3^2 = 9$.

Ex. 2. Let $\sqrt{x} = 2$; then $x = 4$.

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* Either of the 2d and 3d Rules might be conſidered as comprehending both: for multiplication by a is diviſion by $\frac{1}{a}$, and diviſion by a is multiplication by $\frac{1}{a}$.

Ex. 3. Let $\sqrt[3]{ax+x} = b-c$ then $ax+x = \overline{b-c}\rceil^{3} = b^{3} - 3b^{2}c + 3bc^{2} - c^{3}$.

228. Cor. Any equation may be cleared of a ſingle radical quantity, by bringing it to ſtand alone on one ſide of the equation, and then raiſing both ſides to the power whoſe index denominates the root.

Ex. Let $a - \sqrt{bx} = c$ then, by tranſpoſition, $\sqrt{bx} = a - c$, and $bx = \overline{a-c}\rceil^{2}$

229 Rule 5th. *If the ſame root be extracted on both ſides, the ſides will ſtill be equal.*

Ex. Let $x^{2} = 81$, then $x = \sqrt{81} = 9$.

230. If there be two equations, one ſide of one added to one ſide of the other, is equal to the ſum of the remaining ſides, or a ſide of one ſubtracted from a ſide of the other is equal to the remaining ſide of the one ſubtracted from the remaining ſide of the other.

Thus, if $a + b = c$ and $x = dy$, $a + b + x = c + dy$; and $a + b - x = c - dy$.

The former is called adding the equations, and the latter, ſubtracting them.

231. 6th. Any quantity, of which only the firſt power is found in a propoſed equation, may, by the above rules, be made to ſtand alone on one ſide, the other ſide being a value of it in terms of

the other quantities that are concerned in the given equation.

Ex. 1ſt. Given $ax=x+b$, Required the value of x in terms of a and b.

By tranſpoſition $ax-x=b$ or, $\overline{a-1}.\ x=b$.

$\therefore$ Dividing each ſide by $\overline{a-1}$, $x=\frac{b}{a-1}$.

Ex. 2d. Given $ma^2 + b^2 = b^2c - na^2$. Required the value of a^2 in terms of m, b, c, and n.

$ma^2 + na^2 = b^2c-b^2$. or $\overline{m+n}.\ a^2 = b^2c-b^2$.

$\therefore$ Div. by $m+n$, $a^2 = \frac{b^2c-b^2}{m+n}$.

Ex. 3d. Given $xy-xz = ab$.

Required the value of x in terms of y, z and ab.

$$\overline{y-z}.\ x = ab.$$

$$\therefore x = \frac{ab}{y-z}.$$

Ex. 4th. Given $\sqrt{xy}-a = \sqrt{rx}$.

Required the value of $\sqrt{x}$ in terms of y, a and r.

$$\sqrt{xy} - \sqrt{rx} = a$$

or $\sqrt{x}\times\sqrt{y} - \sqrt{x}\times\sqrt{r} = \overline{\sqrt{y}-\sqrt{r}}\times\sqrt{x} = a$.

$$\therefore \sqrt{x} = \frac{a}{\sqrt{y}-\sqrt{r}}.$$

132. The above article includes the ſolution of all ſimple equations whatſoever, which, it is evident,

may always be obtained by means of the following proceſs.

1ſt. Clear the equation of fractions, by art. 224; 2dly. Tranſpoſe all the terms that involve the unknown quantity to one * ſide of the equation, and the known quantities to the other, by art. 221; 3dly. Divide both ſides by the coefficient, or ſum of the coefficients of the unknown quantity, (i. e. by that ſimple or compound expreſſion, which multiplied into the unknown quantity, produces that ſide of the equation where the unknown quantity is found) and the value required is obtained.

In many caſes it will be beſt to tranſpoſe firſt, and then clear off the fractions. (See Ex. 2d.)

EXAMPLES of the reſolution of Simple Equations.

Ex. 1ſt. Let $3x+5=x+9$; required the value of x.

Tranſpoſing x, $3x-x=9-5$, or $2x=4$.

$\therefore$ div. by 2 $\quad x=2$.

Ex. 3d. Let $5x-\frac{5x}{2}+12=\frac{4x}{3}+26$.

by tranſp. $\quad 5x-\frac{5x}{2}-\frac{4x}{3}=26-12=14$.

Mult. by 2×3 $\quad 30x-15x-8x=84$, or $7x=84$.

$\therefore$ div. by 7 $\quad x=\frac{84}{7}=12$.

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* It will be moſt convenient to tranſpoſe the unknown quantity to that ſide which makes the ſum of its coefficients poſitive.

Ex. 3. Let $a - \frac{x-b}{7} = ax - c^2$.

$7a - \overline{x-b} = 7ax - 7c^2$, or $7a - x + b = 7ax - 7c^2$.

$7a + b - 7c^2 = 7ax + x = x \times \overline{7a+1}$.

$\therefore x = \frac{7a+b-7c^2}{7a+1}$.

Examples of Equations that may be reduced to ſimple equations by multiplication or diviſion.

Ex. 4th. Given $\frac{5}{x} + \frac{9}{4} = 16$.

Mul. by x and $5 + \frac{9x}{4} = 16x$, a ſimple equation.

Hence $20 + 9x = 64x$, and $20 = 55x$

$\therefore x = \frac{20}{55} = \frac{4}{11}$.

Ex. 5th. Given $6ax^3 - 12abx^2 = 3ax^3 + 6a^2x^2$

div. by $3ax^2$, $2x - 4b = x + 2a$ a ſimple equation.

Hence $2x - x = 2a + 4b$, or $x = 2a + 4b$.

Ex. 6th. Given $\sqrt{x} + \sqrt{a+x} = \frac{2a}{\sqrt{a+x}}$.

Mul. by $\sqrt{a+x}$, and $\sqrt{x} \times \sqrt{a+x} + \sqrt{a+x} \times \sqrt{a+x} = 2a$

or $\sqrt{x} \times \sqrt{a+x} + a + x = 2a$.

$\therefore \sqrt{x} \times \sqrt{a+x} = a - x$

and ſquaring both ſides $x \times \overline{a+x} = \overline{a-x}|^2$,

or $ax + x^2 = a^2 - 2ax + x^2$.

$\therefore ax = a^2 - 2ax$, and $3ax = a^2$.

$\therefore x = \frac{a^2}{3a} = \frac{a}{3}$.

Ex. 7th. Let $x - \frac{x}{3} = x + 1$.

Subtracting x from both ſides $-\frac{x}{3} = 1$.

Mult. by -3 $x = 1 \times -3 = -3$.*

133. If there be two unknown quantities, their values may be found, provided there be given two independent equations involving them, by any of the three following methods:

1ſt *Method.* If either of the unknown quantities have the ſame coefficient in both equations, ſubtract or add the equations according as the ſign of the unknown quantity is alike or different in the two caſes. In the reſulting equation but one of the unknown quantities will be found, whoſe value may therefore be obtained by the preceding rules.

If neither of the unknown quantities have the ſame coefficient in both the given equations, deduce from them two equations in which one of the unknown quantities ſhall have the ſame coefficients, which may be always done by multiplying the terms of the firſt equation by the coefficient of one of the unknown quantities in the ſecond, and the terms of the ſecond equation by the coefficient of the ſame unknown quantity in the firſt equation. But if the two coefficients have a common factor, it will be beſt firſt to divide them by it, and multiply by the reſpective quotients.

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* All that appears from this concluſion is, that if -3 be ſubſtituted according to the algebraic rules for x in the given equation, the numerical value of each ſide will be the ſame. The equation will be $-3 - \frac{-3}{3} = -3 + 1$, or $-3 + 1 = -3 + 1$.

Having found the value of one of the unknown quantities, the other may be obtained either by the ſame method, or by ſubſtituting this value inſtead of the unknown quantity in one of the given equations, which will reduce it to a ſimple equation involving only one unknown quantity.

2d *Method.* Find two expreſſions for one of the unknown quantities, one from each equation, (art.231.) put theſe expreſſions equal to each other, and the reſulting equation will involve only one unknown quantity, whoſe value may therefore be found, and hence by ſubſtitution the value of the other unknown quantity.

3d *Method.* From either of the two equations find the value of one of the unknown quantities in terms of the other quantities, (art. 231,) and ſubſtitute this value inſtead of that unknown quantity in the other equation, which will then contain only one unknown quantity, whoſe value may therefore be found, and hence the value of the other may be found.

An Example to the three different methods.

Given $\left\{\begin{array}{l} 2x + 3y = 23 \\ 5x - 2y = 10 \end{array}\right\}$ to find the values of x and y.

1ſt Method. Multiplying the
1ſt equation by 5, $10x + 15y = 115$
2d by 2, $10x - 4y = 20$
by ſubtraction $19y = 95$
dividing by 19 $y = \frac{95}{19} = 5.$

Hence $2y = 10$ $\therefore$ from the ſecond equation $5x - 10 = 10$, and (by tranſp.) $5x = 20$

$$\therefore x = \frac{20}{5} = 4.$$

2d Method. From the firſt equation

$$2x = 23 - 3y, \text{ and } x = \frac{23-3y}{2}.$$

from the ſecond, $5x = 10 + 2y$, and $x = \frac{10+2y}{5}$

$\therefore \frac{10+2y}{5} = \frac{23-3y}{2}$, and multiplying by 2×5,

$$20 + 4y = 115 - 15y.$$

by tranſp. $15y + 4y = 115 - 20$, or $19y = 95$.

$\therefore y = \frac{95}{19} = 5.$
Hence $x = 4.$

3d Method. From the firſt equation $x = \frac{23-3y}{2}$. Hence, by ſubſtitution the ſecond equation becomes $5\times\frac{23-3y}{2} - 2y = 10.$

multiplying by 2, $\overline{5\times23 - 3y} - 4y = 20$,
or $115 - 15y - 4y = 20$
by tranſp. $115 - 20 = 15y + 4y = 19y.$
Hence as before $y = 5$ and $x = 4.$

Ex. 2d. Given $\left\{\begin{array}{l} x+y=300 \\ 11x-9y=0 \end{array}\right\}$

2d Method. From the firſt equation

$x = 300 - y$, and $11x = 3300 - 11y$.

from the ſecond, $11x = 9y$.

$\therefore 9y = 3300 - 11y$, and $20y = 3300$.

Hence $y = \frac{3300}{20} = 165$,

and $x = 300 - y = 300 - 165 = 135$.

Ex. 3d. Given $\left\{\begin{array}{l} x+y=s \\ x-y=d \end{array}\right\}$

By addition $2x = s + d$

$$\therefore x = \frac{s+d}{2}.$$

By ſubtraction $2y = s - d$

$$\therefore y = \frac{s-d}{2}.$$

Ex. 4th. Given $\left\{\begin{array}{l} \frac{3x-5y}{2} + 3 = \frac{2x+y}{5} \\ 8 - \frac{x-2y}{4} = \frac{x}{2} + \frac{y}{3} \end{array}\right\}$

From the firſt equation $15x - 25y + 30 = 4x + 2y$

$$15x - 4x = 25y + 2y - 30$$

$$\text{or } 11x = 27y - 30$$

$$\text{hence } x = \frac{27y-30}{11}.$$

From the 2d equation $96 - 3x + 6y = 6x + 4y$

$$96 + 6y - 4y = 6x + 3x$$

or $$9x = 2y + 96$$

hence $$x = \frac{2y + 96}{9}$$

$$\therefore \frac{27y - 30}{11} = \frac{2y + 96}{9}$$

$$243y - 270 = 22y + 1056$$

$$243y - 22y = 1056 + 270$$

or $$221y = 1326.$$

$$\therefore y = \frac{1326}{221} = 6.$$

Hence $x = \frac{2y+96}{9} = \frac{12+96}{9} = \frac{108}{9} = 12.$

Ex. 5th. Given $\left\{\begin{array}{l} 6x - y = 3 \\ 9x - 2y = 4 \end{array}\right\}$

Mult. the 1ſt equat. by 3, $18x - 3y = 9$

ſecond by 2, $18x - 4y = 8$

Sub. $y = 1$

Hence $6x - 1 = 3$ and $6x = 4$.

$$\therefore x = \frac{4}{6} = \frac{2}{3}.$$

Ex. 6. Given $\left\{\begin{matrix} ax + by = c \\ dx + ey = f \end{matrix}\right\}$

$$adx + bdy = cd$$
$$adx + aey = fa$$

Subt. $bdy - aey = cd - fa,$

or $\overline{bd - ae}.\ y = cd - fa$

$$\therefore y = \frac{cd - fa}{bd - ae}.$$

alſo $aex + bey = ce$

$$dbx + bey = bf$$

Subt. $\overline{ae - db}.\ x = ce - bf$

$$\therefore x = \frac{ce - bf}{ae - db}.$$

134. If there be *three unknown quantities*, their values may be found, provided there be given *three independent quantities* involving them.

For from two of the given equations there may be deduced an equation involving only two of the unknown quantities, by any of the preceding methods; and from one of the ſame equations and the third there may be deduced an equation involving the *ſame* two unknown quantities. This being done, there will be two equations involving two unknown quantities, whoſe values may therefore be found by the preceding methods; and hence the value of the third unknown quantity.

Ex. 1ſt. Given $\left\{\begin{array}{l} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{array}\right\}$ required the values of x, y and z.

From the 1ſt equation $x=\frac{m-by-cz}{a}$.

2d $x=\frac{n-ey-fz}{d}$.

3d $x=\frac{p-hy-kz}{g}$.

hence $\frac{m-by-cz}{a}=\frac{n-ey-fz}{d}$.

and $\frac{m-by-cz}{a}=\frac{p-hy-kz}{g}$.

$\therefore md-bdy-cdz=na-eay-faz$

and $mg-bgy-cgz=pa-hay-kaz$

by tranſpoſition $eay-bdy=cdz-faz+na-md$

$$hay-bgy=cgz-kaz+pa-mg$$

$$y=\frac{cdz-faz+na-md}{ea-bd}.$$

$$\therefore y=\frac{cgz-kaz+pa-mg}{ha-bg}.$$

$\therefore \overline{cdz-faz+na-md}\times\overline{ha-bg}=\overline{cgz-kaz+pa-mg}\times\overline{ea-bd}$

or $hacdz-ha^2fz+ha^2n-hamd-bgcdz+bgfaz-bgna+bgmd$

$=eacgz-ea^2kz+ea^2p-eamg-bdcgz+bdkaz-bdpa+bdmg$.

Subtract $bgmd-bgcdz$ from both ſides, and divide by a, and

$hcdz-hafz+han-hmd+bgfz-bgn$

$=ecgz-eakz+eap-emg+bdkz-bdp$

and by tr^{n}. $hcdz - hafz + bgfz - ecgz + eakz - bdkz$ $= eap - emg - bdp - han + hmd + bgn$

$$\therefore z = \frac{eap - emg - bdp - han + hmd + bgn}{hcd - haf + bgf - ecg + eak - bdk}$$

* Hence y will be found =

$$\frac{afp - fmg - cdp - kan + kmd + cgn}{kbd - kae + cge - fbg + fah - cdh}$$

and $x = \dfrac{cep - emk - bfp - hcn + hmf + bkn}{afh - cdh + bkd - aek + ceg - bfg}$

The above ſolution proceeds according to the 2d method in art. 133.

According to the 1ſt method, the ſolution muſt be inſtituted as follows:

Subtracting a times the 2d equation from d times the 1ſt, and a times the 3d from g times the 1ſt, we ſhall have

$bdy + cdz - aey - afz = dm - an$, and $bgy + cgz - ahy - akz = gm - ap$, two ſimple equations involving two unknown quantities, whence the values of y and z may be found as before.

According to the 3d method.

From the 1ſt equation $x = \dfrac{m - by - cz}{a}$ which being

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* It is evident that the value of y may be derived from the value of z by ſubſtituting the coefficients of y for the coefficients of z, and the coefficients of z for the coefficients y in that value. For whether x and z be exterminated, or x and y, the final equation muſt be the ſame, *mutatis mutandis*.

ing ſubſtituted for x in the 2d and 3d equations, they become

$$\left\{\begin{array}{l} d\times\dfrac{m-by-cz}{a}+ey+fz=n \\ g\times\dfrac{m-by-cz}{a}+hy+kz=p \end{array}\right\}$$

&c.

Ex. 2d. Given $\left\{\begin{array}{lll} x+y+z & = & 26 \\ x-y & = & 4 \\ x-z & = & 6 \end{array}\right\}$

by ſubtraction $\left\{\begin{array}{l} 2y+z=26-4=22 \\ y-z=6-4=2 \end{array}\right\}$

by addition $3y = 22+2=24$

$$\therefore y = \frac{24}{3}=8$$

and $x=(4+y=)\ 12$

and $z=(x-6=)\ 6$

In this, as in many other examples, the work may be ſhortened by deviating from the general method of ſolution.

For adding the three equations together, $3x=36$. $\therefore x=12$ as before.

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* The ſolution of the 1ſt example may be conſidered as a *general theorem*, in which the letters a, b, c, &c. m, n, p denote the coefficients and known terms of any propoſed equations, involving three unknown quantities.

In Ex. 2d. a, b, c, d, and g are 1, e and k are -1, m is 26, n is 4, p is 6, and f and h are 0.

Hence $z=\dfrac{eap-emg-bdp-han+hmd+bgn}{hcd-haf+bgf-ecg+eak-bdk}$

becomes $z=\dfrac{-6+26-6-0+0+4}{0-0+0+1+1+1}=\dfrac{18}{3}=6.$

In a ſimilar manner may Ex. 6th in Art. 133, be conſidered as a general *theorem* for *two* unknown quantities.

In

235. In a ſimilar manner may the values of any number of quantities be found, provided there be given as many *independent equations* as unknown quantities.

For let there be m equations and m unknown quantities; then there may be deduced $\overline{m-1}$ independent equations involving only $\overline{m-1}$ of the unknown quantities; from theſe $\overline{m-1}$ equations there may be deduced $\overline{m-2}$ independent equations involving only $\overline{m-2}$ of the unknown quantities; and ſo on, till at laſt there ariſe an equation involving only one unknown quantity, whoſe value being obtained, the values of the reſt of the unknown quantities may be had ſucceſſively from the ſteps of the analyſis.

236. It is evident from the preceding articles, that if there be given m equations, the values of no more than m of the letters that are involved in them, can be determined in terms that ſhall involve only the other letters; e. g. if there be given three equations involving the letters a, b, c, d, e, f, the values of a, b and c may be determined in terms of d, e and f; the values of c, d and e may be determined in terms of a, b and f; &c. but the values of a, b, c and d cannot be determined in terms that involve only e and f.

Hence

237. Hence it follows, that if there be not ſo many equations given as unknown quantities, the conditions are not ſufficient to reſtrict the ſolution; but an indefinite variety of numbers will anſwer the propoſed conditions. For ſelecting from the unknown quantities ſo many as there are equations given, the values of the reſt may be aſſumed *ad libitum;* and this aſſumption, together with the conditions of the given equations, will determine the values of the ſelected quantities.

238. In order to obtain values for the unknown quantities, it is neceſſary that the given equations ſhould be *independent* of each other. For ſuppoſe there be given m equations involving m unknown quantities, and that r of theſe equations are dependent on the others, or ſuch as may be derived from them; then ſetting aſide theſe dependent equations, from the other $\overline{m-r}$ equations, the value of $\overline{m-r}$ of the unknown quantities may be determined, if the reſt be aſſumed; and the anſwers (whatever they be) will neceſſarily ſatisfy the conditions of the r dependent equations, becauſe theſe equations require nothing more for their truth, than that the other equations ſhould be juſt. Therefore the values of the m unknown quantities are not in this caſe limited, and conſequently by no reaſoning * whatſoever can parti-

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* Not only the above methods of ſolution, but all others whatſoever muſt fail, in attempting to determine the values of m quantities

cular values be obtained without aſſuming the values of r of them.

239. When there are more equations given than unknown quantities, as $m+r$ equations, and m unknown quantities, the values of the unknown quantities may be determined from any m of the equations; and ſhould their values, thus obtained from different equations be found to differ, the given equations are incongruous, and if the values be found to agree, r of the equations are unneceſſary and dependent on the reſt.

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ties from m equations, that are not all independent. For it would be abſurd, that any reaſoning ſhould determine that which is not reſtricted to a particular determination.

It is ſometimes difficult to diſcover whether equations be dependent or not.

Thus in the equations

$$\begin{array}{r}
x + 3y + 4z = 9 \\
3x - 2y + 17z = 25 \\
x + 14y - z = 11
\end{array}$$

It is not obvious at firſt ſight that the third equation is derived from the other two.

OF THE SOLUTION OF PROBLEMS BY EQUATIONS.

240. EVERY *Equation* that involves one or more unknown quantities may be juſtly conſidered as the ſtatement of a *Problem*, or at leaſt of one of the conditions of a *Problem*, in *Algebraic* language.

Thus the equation $\frac{x}{2}+\frac{x}{3}=5$ is a brief ſtatement of the following problem:

" There is a certain number, the third part of which being added to the half of it, the ſum will be 5."

Solving the equation is inquiring what that number is, and the value found for x is the anſwer to the inquiry.

The number that anſwers the above condition, might be inveſtigated without the help of algebraic ſymbols, by the common principles of reaſoning,* as might the anſwer to any other problem or queſtion whatſoever. But the multiplicity of words would render the proceſs tedious and perplexed.

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* The inveſtigation would be as follows: Since the ſum of a half and a third part of the number is 5, double as much, or the ſum of two halves and two thirds muſt be 10, but two halves of a number is the whole of it; therefore the whole of the number together with two thirds of it makes up 10; and three times as much as this muſt be 30, i. e. three times the number, together with three times two thirds of it makes up 30; but three times two thirds of a number, or ſix thirds, is twice the whole of it; therefore three times the number together with twice the number makes up 30, or five times the number is 30, therefore the number itſelf is a 5th part of 30, which is 6.

To take another example; the two equations $x+y=9$ and $x-y=1$ are nothing more than a ſtatement of the following problem.

"There are two numbers whoſe ſum is 9, and whoſe difference is 1."

241. It appears from the preceding articles, that if a problem be properly limited, or the quantities required be determinable from the propoſed conditions, there will be as many independant conditions as quantities ſought. Hence, if a ſymbol be put to denote each of the quantities ſought, there will be as many independent equations expreſſive of the conditions of the problem as there are unknown quantities, the values of which may therefore be obtained. But in many caſes, inſtead of ſubſtituting for each of the quantities ſought, it will be more convenient to ſubſtitute for *one* only, and from the conditions of the problem, to infer names or expreſſions for the others, in terms of that one and of known quantities; and becauſe the number of conditions will be one more than the number of quantities which muſt thus have expreſſions derived for them, there will at laſt remain a condition, which will not have been employed, to be ſtated in an equation; and from this equation the value of the unknown quantity may be

obtained,

obtained, and hence the values of the expreſſions for the other unknown quantities.

By this method the ſolution is begun one or more ſteps forwarder than it would be if each of the unknown quantities had a letter ſubſtituted for it, and unneceſſary trouble is thereby avoided.

To make this ſtill plainer by an example, let the above-mentioned problem be reſumed, viz. To find two numbers, whoſe ſum is 9, and difference 1. Here if we put x to denote the greater of the two numbers, and y to denote the leſs, we ſhall have $x+y=9$ and $x-y=1$; hence $x=9-y$; which value being ſubſtituted for x in the ſecond equation, it becomes $9-y-y=1$. But if y be put for the leſs of the two numbers, it readily appears that the greater may be expreſſed by $9-y$, (for if the leſs be ſubtracted from the ſum, there will remain the greater) and then from the other condition of the problem is had the equation $9-y-y=1$ as before. And becauſe from the above two equations $x+y=9$ and $x-y=1$, two values may be had for either of the two quantities x and y, one from each of the equations, there will be four methods, by which we may ſubſtitute for one of the unknown quantities, and derive an expreſſion for the other. For,

1ſt. We may ſubſtitute for the *greater*, and derive an expreſſion for the leſs from the given ſum,——by putting the two quantities to be x, and $9-x$; or

2dly. We may ſubſtitute for the *greater* and derive an expreſſion for the leſs, from the given difference,——by putting the two quantities to be x and $x-1$; or

3dly. We may ſubſtitute for the *leſs*, and derive an expreſſion for the greater, from the given ſum,——by putting the two quantities to be y, and $9-y$; or

4thly. We may ſubſtitute for the *leſs*, and derive an expreſſion for the greater, from the given difference,——by putting the two quantities to be y and $y+1$.

And a ſimilar choice of expreſſions will preſent itſelf in other inſtances, in ſelecting from which, the only thing to be conſidered is, which of the expreſſions will make the equation the moſt ſimple and ready to be ſolved. But ſkill in this can only be acquired by practice.

Problems producing Simple Equations.

PROBLEM 1ſt.

What number is that, to the double of which if 16 be added, the ſum will be 188?

Let x denote the required number; then $2x$ will denote the double thereof; and by the queſtion $2x+16=188$.

Hence by tranſpoſition $2x = 188 - 16 = 172.$

$\therefore$ div. by 2 $x = \frac{172}{2} = 86$, the number required.

PROBLEM 2d.

A trader allows 100 l. per annum for the expences of his family; and augments yearly that part of his ſtock, which is not ſo expended, by a third part of itſelf. At the end of three years his original ſtock was doubled: What had he at firſt?

Let his 1ſt ſtock be	x
Of which ſpending 100l. the 1ſt year, there remains	$x - 100$
This rem^r. is increaſed by a 3d of itſelf, and becomes	$x - 100 + \frac{x-100}{3} = \frac{4x-400}{3}$
The 2d year he ſpends 100l. and there remains	$\frac{4x-400}{3} - 100 = \frac{4x-700}{3}$
This rem^r. is increaſed by a 3d of itſelf, and becomes	$\frac{4x-700}{3} + \frac{4x-700}{9} = \frac{16x-2800}{9}$
The 3d year he ſpends 100l. and there remains	$\frac{16x-2800}{9} - 100 = \frac{16x-3700}{9}$
This rem^r. is increaſed by a 3d of itſelf, and becomes	$\frac{16x-3700}{9} + \frac{16x-3700}{27} = \frac{64x-14800}{27}$
But per queſtion at the end of the 3d year his original ſtock is doubled $\therefore$	$\frac{64x-14800}{27} = 2x$

Hence $64x - 14800 = 54x$

and $10x = 14800$

$\therefore$ $x = 1480$

Therefore his ſtock was 1480l.

PROBLEM 3d.

Required what number that is, which being divided into three equal parts and into four equal parts, the product of the three parts is equal to the product of the four parts

Let x denote the number;

Then the three equal parts are each of them $\frac{x}{3}$; and the four equal parts are each of them $\frac{x}{4}$; therefore by the queſtion

$$\frac{x}{3} \times \frac{x}{3} \times \frac{x}{3} = \frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} \times \frac{x}{4}$$

and div. by x^3 $\frac{1}{3^3} = \frac{x}{4^4}$

$$\therefore x = \frac{4^4}{3^3} = \frac{256}{27} = 9\frac{13}{27}.$$

PROBLEM 4th.

A gentleman diſtributing money, found he wanted 10s. to be able to give 5s. to each perſon; therefore he gives to each 4s. only, and finds he has 5s. left. Required the number of perſons and ſhillings?

Let x denote the number of perſons; then ſince with 10s. more there would have been $5x$ ſhillings, the number of ſhillings is $5x - 10$.

And ſince with 5s. leſs there would have been $4x$ ſhillings, the number of ſhillings is $4x + 5$.

$\therefore 5x - 10 = 4x + 5$,
and $x = 15$.

The number of perſons therefore is 15, and the number of ſhillings is $(4x + 5 =)\ 65$.

PROBLEM 5th.

What number is that, which being ſeverally added to 36 and to 52, the former ſum will be three fourths of the latter?

Let the number be denoted by x.

Then the former ſum is $36+x$.
and the latter ſum is $52+x$.

$$\therefore 36 + x = \frac{3}{4} \times \overline{52 + x}$$

and $4 \times \overline{36 + x} = 3 \times \overline{52 + x}$, or $144+4x=156+3x$
whence $x = 12$

$\therefore$ if 12 be added to 36 and to 52, the former ſum (48) will be $\frac{3}{4}$ths of the latter ſum (64.)

PROBLEM 6th.

What number is that, which being ſeverally added to 36 and to 52, the former ſum will be two thirds of the latter?

Purſuing the ſame method as in the laſt problem,

$36 + x = \frac{2}{3} \times \overline{42 + x}$ 52

and $3 \times \overline{36 + x} = 2 \times \overline{52 + x}$, or $108 + 3x = 104 + 2x$ whence $x = -4$.

$\therefore$ if -4 be * added (algebraically) to 36 and to 52, the former ſum (32) will be $\frac{2}{3}$ds of the latter ſum (48.)

PROBLEM

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* If there be two numbers, a, and $\overline{a+b}$, and a be m $n$ths of $a+b$, and the ſame quantity (d) be added to each, the former ſum $(a+d)$ will be greater than m $n$ths of the latter ſum $(a+b+d)$. For, (if a be $\frac{m}{n}$ths of $a+b$) when d is added to $a+b$, only $\frac{m}{n}$ths of d muſt be added to a, in order that the latter ſum may be $\frac{m}{n}$ths of the former.

And converſely, if the ſame quantity (d) be ſubtracted from each, $a-b$ will be leſs than $\frac{m}{n}$ ths of $a+b-d$.

Hence

PROBLEM 7th.

A courier paſſing through a certain place P, travels at the rate of five miles in two hours. Four hours after, another paſſes through the ſame place travelling the ſame way at the rate of ſeven miles in two hours. How long and how far muſt they travel before the firſt be overtaken by the ſecond?

Let x denote the number of hours the firſt courier muſt travel after paſſing through P before he be overtaken; then $x-4$ will be the number of hours the ſecond muſt travel. And becauſe the firſt travels $\frac{5}{2}$ miles in one hour, he will travel $\frac{5}{2}\times x$ miles in x hours; and becauſe the ſecond travels $\frac{7}{2}$ miles in one hour, he will travel $\frac{7}{2}\times\overline{x-4}$ miles in $\overline{x-4}$ hours; but the firſt in x

Hence it appears, that becauſe 36 is *greater* than two thirds of 52, it is not by the addition of any quantity to each, but by the ſubtraction, that the former ſum can become two thirds of the latter, and the true equation is $36 - x = \frac{2}{3}\times\overline{52-x}$.

But if $+ x$ be conſidered as denoting either a poſitive or a negative quantity, as occaſion may require, $36 + x$ may be put equal to $\frac{2}{3}\times\overline{52+x}$, or *generally*, $a + x$ may be always put equal to $\frac{c}{d}\times\overline{a+b+x}$.

hours, and the ſecond in $x-4$ hours, travel over the ſame ground,—viz. from *P* to the point where the firſt is overtaken.

Hence, $\frac{5x}{2} = \frac{7}{2}\times\overline{x-4} = \frac{7x-28}{2}$,

and $5x = 7x - 28$; whence $x = 14$, the number of hours the firſt muſt travel; and $x-4 = 10$ the number of hours the ſecond muſt travel. Alſo the number of miles travelled over being $\frac{5x}{2}$ is 35.

If the ſecond courier had travelled *ſlower* than the firſt, the * queſtion muſt have been, How far back, *previous* to the arrival of the firſt at *P*, *did* the firſt overtake the other.

E e PROBLEM

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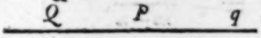
* The ſame method of ſolution as is employed above will ſerve for this caſe, provided that the anſwer, when it is found to be negative, be conſidered as denoting how long *back* it was *ſince* the couriers were together. For, to make the problem *general*, Let two bodies *A* and *B* move in the ſame line *Qq* and in the ſame direction,



from *Q* towards *q*, *A* with a velocity of *m* miles in an hour, and *B* with a velocity of *n* miles in an hour, and let *A* paſs through *P* *d* hours before *B*, then,

CASE 1ſt.

If *n* be greater than *m*, the direct ſolution is ſimilar to that of Problem 7, and the equation will be $mx = n\times\overline{x-d}$ and $x = \frac{nd}{n-m}$. But

CASE 2d.

If *m* be greater than *n*, the bodies *were* together at ſome point *Q*, *previous* to their paſſing through *P*, and the direct ſolution is as follows:

Let

PROBLEM 8th.

A hare 50 of her leaps before a greyhound, takes four leaps to the greyhound's three; but the greyhound in two leaps goes over as much ground as the hare in three. How many leaps muſt the greyhound take to catch the hare?

Let x denote the number of hours elapſed from the time of their being together at Q, to the time at which A paſſes through P; then $x+d$ is the number of hours elapſed from the time of their being together to the time at which B paſſes through P; and becauſe in one hour A moves m miles, in x hours A moves mx miles, the ſpace QP; and becauſe in one hour B moves n miles, in $\overline{x+d}$ hours B moves $n.\overline{x+d}$ miles, the ſame ſpace QP.

$$\therefore mx = nx + nd$$

$$\text{and } x = \frac{nd}{m-n}.$$

Now, if it be undetermined which of the two, m and n, be the greater, and the problem be *conſidered* as belonging to the 1ſt Caſe, the ſolution $x = \frac{nd}{n-m}$ will, if m be greater than n, be negative, and denote the time elapſed *ſince* the bodies were together at Q; for $\frac{nd}{n-m} = -\frac{nd}{m-n}$.

The ſpace from P to the point where the two bodies are together, is in both caſes denoted by mx, which in the 1ſt Caſe, where n is greater than m, is $m\times\frac{nd}{n-m}$; and in the 2d Caſe it is $m\times\frac{nd}{m-n}$. i. e. if q and Q be the reſpective points, $m\times\frac{nd}{n-m} = Pq$, and $m\times\frac{nd}{m-n} = PQ$, therefore in both caſes it may be ſaid to be $m\times\frac{nd}{n-m}$, if it be meaſured from P in the direction of motion when this expreſſion is poſitive, and in the contrary direction, when this expreſſion is negative.

Let x denote the number of leaps the greyhound muſt take: then if the greyhound took *one* leap while the hare took four, the hare would have $4x$ leaps to take; but the greyhound takes three while the hare takes four; therefore the hare has $\frac{4x}{3}$ leaps to take. Hence, in going over the whole ground that the greyhound paſſes over, the hare would have to take $\overline{50+\frac{4x}{3}}$ leaps. But by hypotheſis the greyhound, in going over the ſame ground, will take but $\frac{2}{3}$ of the number of leaps that the hare takes.

$$\therefore\ x = \frac{2}{3}\times\overline{50+\frac{4x}{3}} \quad\text{or}\quad 3x = 100 + \frac{8x}{3}$$

whence $x = 300$.

Problem 9th.

If A and B together can perform a piece of work in a days, A and C together in b days, and B and C together in c days, in what time will each of them perform it alone?

Let A perform the work in x days, B in y days, and C in z days; then

A performs $\frac{1}{x}$ of the work in a day, and $\frac{a}{x}$ of it in a days. B performs $\frac{1}{y}$ of the work in one day, and $\frac{a}{y}$ of it in a days.

$\therefore$ An $\overline{\frac{a}{x}+\frac{a}{y}}$ part of the work is the whole of it, or $\frac{a}{x}+\frac{a}{y}=1$.

By a ſimilar reaſoning it appears, that

A performs $\frac{b}{x}$ of it in b days,

C performs $\frac{b}{z}$ of it in b days,

$\therefore \frac{b}{x}+\frac{b}{z}=1$.

Alſo B performs $\frac{c}{y}$ of it in c days,

and C performs $\frac{c}{z}$ of it in c days,

$\therefore \frac{c}{y}+\frac{c}{z}=1$.

Hence the three following equations;

$$ay+ax=xy$$
$$bz+bx=xz$$
$$cz+cy=yz$$

From the 1ſt $x=\frac{ay}{y-a}$.

From the 3d $z=\frac{cy}{y-c}$.

which values being ſubſtituted in the 2d equation, it becomes $\frac{bcy}{y-c}+\frac{bay}{y-a}=\frac{acy^2}{\overline{y-c}\times\overline{y-a}}$

div. by y and $\frac{bc}{y-c}+\frac{ba}{y-a}=\frac{acy}{\overline{y-c}\times\overline{y-a}}$

Hence, $\frac{bc\times\overline{y-a}+ba\times\overline{y-c}}{\overline{y-c}\,.\,\overline{y-a}}=\frac{acy}{\overline{y-c}\,.\,\overline{y-a}}$

and $bcy-abc+bay-abc=acy$,

whence $y=\frac{2abc}{bc+ba-ac}$

hence, $z=\frac{cy}{y-c}=\frac{2abc^2}{bc+ba-ac}\div\overline{\frac{2abc}{bc+ba-ac}-c}$

$=\frac{2abc^2}{bc+ba-ac}\div\frac{abc-bc^2+ac^2}{bc+ba-ac}=\frac{2abc^2}{abc-bc^2+ac^2}$

$=\frac{2abc}{ab+ac-bc}$ and $x=\frac{ay}{y-a}=\frac{2abc}{ac+bc-ab}$.*

PROBLEM

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* It appears from the ſolution that *a*, *b* and *c* muſt be ſo aſſumed, that of the three products *ab*, *ac*, *bc*, the ſum of any two ſhall be greater than the third; otherwiſe the values of *x*, *y* and *z* will not all be poſitive.

And the ſame reſtriction for their values may be deduced from the nature of the queſtion. For *B* alone could not perform the whole work in *a* days; therefore *B* could not perform an *a*th part in one day. In the ſame manner *C* could not perform a *b*th part in one day. Hence *B* and *C* together could not perform a $\overline{\frac{1}{a}+\frac{1}{b}}$ part in one day;

therefore

PROBLEM 10th.

Required what number it is, that conſiſts of three places, and has the following properties: 1ſt. Its digits have a common difference. 2d. If it be divided by the ſum of its digits the quotient will be 48. 3d. If 198 be ſubtracted from it, the remainder will be expreſſed by the ſame three digits in an inverted order.

therefore they could not perform a $c \times \overline{\frac{1}{a}+\frac{1}{b}}$ part in c days; but, by ſuppoſition, they perform the whole in c days; therefore a $c \times \overline{\frac{1}{a}+\frac{1}{b}}$ part is greater than the whole; or $c \times \frac{b+a}{ab}$ is greater than 1, therefore $ca+cb$ is greater than ab. And in a ſimilar manner might it be ſhewn, that $ab+bc$ is greater than ac, and $ab+ac$ greater than bc.

If a, b, and c be ſo aſſumed, that the denominator of one of the expreſſions for x, y and z is 0, it implies that one of the agents did nothing. If they be ſo aſſumed, that one of the expreſſions becomes negative, it implies that one of the agents, inſtead of promoting the work, either obſtructed it, or undid it to a certain extent. For let the expreſſion for x, $\frac{2abc}{ac+bc-ab}$, be negative; then the poſitive value of this expreſſion, or $\frac{2abc}{ab-ac-bc}$, is the ſame value as would be found for x from the three equations $\frac{a}{y}-\frac{a}{x}=1$, $\frac{b}{z}-\frac{b}{x}=1$, $\frac{c}{y}+\frac{c}{z}=1$; but theſe equations are derived from the following problem.

If B obſtructed by A can perform a piece of work in a days, and C obſtructed by A in b days, and B and C together in c days, in what times can B and C perform it, and A undo it, each acting ſeparately?

Let y denote the difference of the digits, and x denote the middle digit; then the three digits are $x+y$, x and $x-y$, and their ſum is $3x$; and becauſe the diminution of the number inverts the order of the digits, the greateſt digit, $x+y$, ſtands in the place of hundreds.

$\therefore$ the number is $100\times\overline{x+y} + 10x + x-y = 111x + 99y$,

and the digits inverted form the number $100\times\overline{x-y} + 10x + x + y = 111x - 99y$,

Hence, $\dfrac{111x + 99y}{3x} = 48$

and $111x + 99y - 198 = 111x - 99y$.

From the laſt equation $198y = 198 \quad \therefore y = 1$;

Hence, $\dfrac{111x + 99}{3x} = 48 \quad \therefore 111x = 3\times 48x - 99$

and $33x = 99$, whence $x = 3$.

$\therefore$ the digits are 4, 3, 2, and the number required is 432.

The order of the digits from higheſt to loweſt might have been aſſumed $x-y$, x, $x+y$, and the ſame concluſion would have been obtained with equal facility; for then y would have been found $= -1$.

CHAP.

CHAPTER VIII.

OF QUADRATIC EQUATIONS.

242. EQUATIONS are either *pure* or *affected.* A *pure equation* is that, in which only one power of the unknown quantity is found.

An *affected equation* is that, in which different powers of the unknown quantity are found among the ſeveral terms.

Thus, $a^2+ax^2=b^2$, and $ax^2-b^2=m^2+x^2$ are pure equations. And $x^2-ax=b^2$, and $x^4+x^2=17$, are affected equations.

243. Any pure equation (whatever be the dimenſion of it) may be ſolved by the rules of the preceding chapter; for the *power* of the unknown quantity being (by thoſe rules) diſengaged from other quantities, and brought to ſtand alone on one ſide of the equation, the *ſimple* unknown quantity will be equal to the ſquare root, or cube root, or 4th root, &c. of the other ſide of the equation, according as the power of the unknown quantity is the 2d, 3d, 4th, &c. reſpectively.

Thus, if $x^2=ab$, x the ſquare root of one ſide will be equal to $\pm\sqrt{ab}$ the ſquare root of the other ſide. If $a^4+ax^2=b^4$, by tranſpoſition $ax^2=b^4-a^4$, and $x^2=\frac{b^4-a^4}{a}$, $\therefore x=\overline{\frac{b^4-a^4}{a}}\Big|^{\frac{1}{2}}$

If $ax^m-b=x^m-c$, (where m may denote any number whatſoever integral or fractional)

$$ax^m-x^m=b-c,\text{ and }x^m=\frac{b-c}{a-1},$$

$$\therefore x=\overline{\frac{b-c}{a-1}}\Big|^{\frac{1}{m}}$$

Hence pure equations as to their reſolution may be claſſed with ſimple; and the purpoſe of this chapter is to explain the method of finding the value of the unknown quantity in affected quadratic equations, for which purpoſe it is neceſſary to premiſe the following

244. Lemma. If to any expreſſion of the form x^2+px or x^2-px, conſiſting of two terms, one of which is a poſitive ſquare, and the other the root of that ſquare multiplied into ſome coefficient (p or $-p$) there be added $\left(\frac{p^2}{4}\right)$ the ſquare of half that coefficient, the expreſſion becomes the complete ſquare of a binomial, and its root, or the binomial itſelf, is the ſum or difference of the root of the firſt term and the half coefficient, accord

ing as the ſign of the ſecond term is poſitive or negative.

In other words, if to $x^2 + px$ be added $\frac{p^2}{4}$ the reſult, $x^2 + px + \frac{p^2}{4}$, is the ſquare of $x + \frac{p}{2}$; and if to $x^2 - px$ be added $\frac{p^2}{4}$ the reſult, $x^2 - px + \frac{p^2}{4}$ is the ſquare of $x \sim \frac{p}{2}$.

Examples.

1ſt. If to $x^2 - x$ (or $x^2 - 1 \times x$) there be added $\left.\overline{\frac{1}{2}}\right|^2$ it becomes $x^2 - x + \frac{1}{4}$, whoſe ſquare root is $x \sim \frac{1}{2}$.

2d. If to $x + 2ax^{\frac{1}{2}}$ there be added a^2, it becomes $x + 2ax^{\frac{1}{2}} + a^2$, whoſe ſquare root is $x^{\frac{1}{2}} + a$.

3d. If to $x^{2m} - 36x^m$ there be added 18^2, it becomes $x^{2m} - 36x^m + 18^2$, whoſe ſquare root is $x^m \sim 18$.

4th. If to $x^2y^2 + c^2xy$ there be added $\frac{c^4}{4}$ it becomes $x^2y^2+c^2xy+\frac{c^4}{4}$ whoſe ſquare root is $xy+\frac{c^2}{2}$.

5th. If to $x^2+ax+2\sqrt{x^2+ax}$ there be added 1, it becomes $x^2+ax+2\sqrt{x^2+ax}+1$, whoſe ſquare root is $\sqrt{x^2+ax}+1$.

245. Any affected quadratic equation, whatever be the coefficient of the ſquare of the unknown quantity, may, by dividing the terms by that coefficient, be reduced to an equation in which *unity* ſhall be the coefficient of the ſquare of the unknown quantity; hence any affected quadratic equation, in which the unknown quantity can have any *real poſitive* value, may be reduced to one of the three following forms;

1ſt. $x^2+px=r$
2d. $x^2-px=r$
3d. $px-x^2=r$, in all which x denotes the unknown quantity, and p and r the known quantities; for either the ſum of the two terms x^2 and px, or the exceſs of one of them above the other, may be aſſerted to be equal to a given quantity r.

There is indeed another form of affected quadratic equations, viz. $x^2+px=-r$; but this form can have no place in the *preſent* inquiry

(which

(which is only concerning the real poſitive values of the unknown quantity) becauſe the equation $x^2 + px = -r$ cannot be true, unleſs by x be meant a negative quantity.

246. The 1ſt and 2d of theſe equations are always poſſible, whatever be the magnitude of r; the laſt only when r is not greater than $\frac{p^2}{4}$.

For it is evident, that if in the firſt of theſe equations x be ſuppoſed to increaſe continually from nothing, *ad infinitum*, the quantity $x^2 + px$ will increaſe from nothing at the ſame time, and paſs through all degrees of magnitude whatſoever. Therefore how great or how ſmall ſoever the magnitude of r be taken, there will be *one* certain value of x, and *one only*, which can make $x^2 + px$ equal to it.

In the ſecond of theſe equations it is evident that x muſt be greater than p. Now if x, from being equal to p, be ſuppoſed to increaſe *ad infinitum*, $x^2 - px$ will increaſe at the ſame time from nothing (to which it is equal when x is $= p$) *ad infinitum*, and this continually, or without ever decreaſing. For $x^2 - px$ is $\overline{x-p} \times x$; but as x increaſes both theſe factors increaſe; and conſequently their product, $x^2 - px$; therefore whatever be the magnitude of r, there will be *one*, and *but*

but one value of x, that being ſubſtituted for it, will make $x^2 - px$ equal to r.

In the third of theſe equations, in which px is ſuppoſed to be *greater* than x^2, and the exceſs to be equal to r, it is evident that x muſt be leſs than p. Now if x be ſuppoſed to increaſe from 0, till it be equal to p, $px - x^2$ will, in the ſame time, increaſe from nothing till it arrive at a certain magnitude, and then decreaſe to nothing again. For when x is nothing, $px - x^2$ will be nothing; and when x is equal to p, $px - x^2$ will again be equal to nothing. It muſt therefore have increaſed, and decreaſed again, while x was increaſing from 0 to p. In order to ſhew the greateſt magnitude $px - x^2$ can ever attain, it is neceſſary to obſerve, that if a given quantity be divided into two parts, the product of the two parts will be the greateſt poſſible, when they are equal to each other, and conſequently each equal to half the given quantity, the truth of which may be ſhewn as follows. Let a denote the given quantity; then the two parts are either equal or unequal; if they be equal, they are each of them $\frac{a}{2}$, and their product is $\frac{a^2}{4}$. If they be unequal, one is greater than $\frac{a}{2}$, and the other exactly as much leſs; let $\frac{a}{2} + z$ denote the greater; then

$\frac{a}{2} - z$ will denote the leſs; and their product will be $\frac{a^2}{4} - z^2$; but $\frac{a^2}{4} - z^2$ is leſs than $\frac{a^2}{4}$, i. e. the product of any two unequal parts is leſs than the product of the equal parts. Now $px - x^2$, or $\overline{p-x} \times x$, is the product of two parts $\overline{p-x}$ and x, whoſe ſum is always p, therefore this product is at its maximum when $p - x = x$, i. e. when x and $p - x$ are each equal to $\frac{p}{2}$; but the product then is $\frac{p^2}{4}$; therefore r may not be greater than $\frac{p^2}{4}$.

Moreover, if $r = \frac{p^2}{4}$, x will have *one*, and *but one* value,—viz. $\frac{p}{2}$; but if r be leſs than $\frac{p^2}{4}$, p is divided into two unequal parts, $p - x$ and x, of which $p - x$ may be the greater, and then x is leſs than $\frac{p}{2}$; or x may the greater, and then x is greater than $\frac{p}{2}$; therefore if r be leſs than $\frac{p^2}{4}$, x has two values, the one greater, and the other leſs than $\frac{p}{2}$.

The preceding equations are reſolved in the following manner. In the equation $x^2 + px = r$; the value of x is found to be $\frac{\sqrt{p^2 + 4r} - p}{2}$ by adding $\frac{p^2}{4}$ (the ſquare of $\frac{p}{2}$, or of half the coefficient of the ſecond term) to both ſides of the equation, then extracting the ſquare root of both ſides, and tranſpoſing the terms. For ſince $x^2 + px = r$,

$$x^2 + px + \frac{p^2}{4} = r + \frac{p^2}{4} = \frac{p^2 + 4r}{4}$$

$$\therefore \sqrt{x^2 + px + \frac{p^2}{4}} = \sqrt{\frac{p^2 + 4r}{4}};$$ i. e. $x + \frac{p}{2}$ $\frac{\sqrt{p^2 + 4r}}{2}$ and $x = \frac{\sqrt{p^2 + 4r} - p}{2}$.

In the equation $x^2 - px = r$, the value of x is found, by the ſame proceſs, to be $\frac{\sqrt{p^2 + 4r} + p}{2}$. For adding $\frac{p^2}{4}$ to both ſides, $x^2 - px + \frac{p^2}{4} = \frac{p^2}{4} + r = \frac{p^2 + 4r}{4}$. ∴ the roots of theſe quantities are equal, i. e. the difference of x and $\frac{p}{2}$ is equal to $\frac{\sqrt{p^2 + 4r}}{2}$, but this difference is $x - \frac{p}{2}$ and not $\frac{p}{2} - x$, becauſe x is greater than $\frac{p}{2}$; $\therefore x - \frac{p}{2} = \frac{\sqrt{p^2 + 4r}}{2}$ and $x = \frac{\sqrt{p^2 + 4r} + p}{2}$.

In the equation $px - x^2 = r$, the proceſs is ſomewhat different: For here $px - x^2$, and its equal, r, are each of them to be ſubtracted from $\frac{p^2}{4}$, than which, it has been ſhewn, they are always leſs:

$\therefore \frac{p^2}{4} - \overline{px - x^2} = \frac{p^2}{4} - r$, or $\frac{p^2}{4} - px + x^2 = \frac{p^2 - 4r}{4}$; therefore the roots of theſe quantities are equal; or the difference of x and $\frac{p}{2}$ is equal $\frac{\sqrt{p^2 - 4r}}{2}$; i. e. if x be leſs than $\frac{p}{2}$, $\frac{p}{2} - x = \frac{\sqrt{p^2 - 4r}}{2}$; and if x be greater than $\frac{p}{2}$, $x - \frac{p}{2} = \frac{\sqrt{p^2 - 4r}}{2}$; therefore in the firſt caſe we ſhall have $x = \frac{p - \sqrt{p^2 - 4r}}{2}$, and in the ſecond caſe, $x = \frac{p + \sqrt{p^2 - 4r}}{2}$.

When $\frac{p^2}{4} = r$, $p^2 = 4r$, $\frac{p^2 - 4r}{4} = 0$, and $\frac{p - \sqrt{p^2 - 4r}}{2} = \frac{p + \sqrt{p^2 - 4r}}{2} = \frac{p}{2}$; agreeably to what was obſerved before.

If, by changing the ſign of each term in this 3d form, it be put $x^2 - px = -r$, the proceſs will be the ſame as in the other two,—viz. by adding $\frac{p^2}{4}$ to each ſide.

246. The above inveſtigation ſhews what real quantities ſubſtituted for x will make the equations true. And it appears that in two of the claſſes of quadratic equations, viz.
$x^2 + px = r$ and $x^2 - px = r$ there can be but one poſitive quantity, which may be ſubſtituted for x; but that in the other claſs, viz. $px - x^2 = r$, there will be two, provided r be not ſo great as $\frac{p^2}{4}$; and none at all if r be greater than $\frac{p^2}{4}$. But if negative quantities, and impoſſible quantities be allowed to be ſubſtituted as values of the unknown quantity, then in every quadratic equation the unknown quantity will have two values, whatever be the magnitude of p and r, and whatever be the ſigns of the terms.

For any affected quadratic equation whatſoever may (by changing the ſigns of the terms when x^2 is given negative) be reduced to one of theſe four forms, $x^2 + px = r$, $x^2 - px = r$, $x^2 - px = -r$, $x^2 + px = -r$. In each of which, if $\frac{p^2}{4}$ be added to both ſides, the ſide involving x will become a complete ſquare, whoſe two roots will be $x + \frac{p}{2}$,

and

and $-x-\frac{p}{2}$ when px is poſitive, and $x-\frac{p}{2}$ and $\frac{p}{2}-x$ when px is negative; and in each caſe* either of the roots may be put equal both to the poſitive and to the negative root of the other ſide; whence, by tranſpoſition, two values of x will ariſe.

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[*] It is not by putting one of the roots of the ſide involving x equal to one of the roots of the known ſide, and the other root equal to the other root of the known ſide that two values of x ariſe, but by putting one root of the ſide involving x equal to *each* of the roots of the other ſide. Thus, in the 1ſt form, it is not by putting $x+\frac{p}{2}=+\frac{\sqrt{p^2+4r}}{2}$ and $-x-\frac{p}{2}=-\frac{\sqrt{p^2+4r}}{2}$ that the two values of x are obtained, for theſe two equations would give one and the ſame value of x; but by putting either $x+\frac{p}{2}$ or $-x-\frac{p}{2}$ (it makes no difference which) equal both to $+\frac{\sqrt{p^2+4r}}{2}$ and to $-\frac{\sqrt{p^2+4r}}{2}$.

The equation $x+\frac{p}{2}=-\frac{\sqrt{p^2+4r}}{2}$ is equally juſt as the equation $x+\frac{p}{2}=+\frac{\sqrt{p^2+4r}}{2}$; becauſe in *this* view of quadratic equations, the ſymbol x or $+x$ denotes any quantity (whether poſitive or negative) which the equation may require for its juſtification.

247. In order that the values of the two roots in the different forms may be more clearly underſtood, the ſolutions of each of the *four general equations* are here ſubjoined and exemplified.

FIRST FORM.

Let $x^2+px=r$; then $x^2+px+\frac{p^2}{4}=\frac{p^2+4r}{4}$

$\therefore x+\frac{p}{2}$ is equal to any quantity, which, multiplied into itſelf, produces $\frac{p^2+4r}{4}$. i. e.

$$x+\frac{p}{2}=+\frac{\sqrt{p^2+4r}}{2}, \text{ or } -\frac{\sqrt{p^2+4r}}{2}$$

$$\therefore x=\frac{\sqrt{p^2+4r}-p}{2} \text{ or } \frac{-\sqrt{p^2+4r}-p}{2},$$ the former of which expreſſions is poſitive, and the latter negative.

Ex. Let $x^2+3x=10$

Then $x^2+3x+\frac{9}{4}=10+\frac{9}{4}=\frac{49}{4}$

$\therefore x+\frac{3}{2}=\pm\sqrt{\frac{49}{4}}=\pm\frac{7}{2}$,

and $x=\pm\frac{7}{2}-\frac{3}{2}=2$, or -5.*

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* If -5 be ſubſtituted for x in the given equation, it becomes $-5\times-5+3\times-5=10$, or $25-15=10$.

SECOND FORM.

Let $x^2 - px = r$; then $x^2 - px + \frac{p^2}{4} = \frac{p^2 + 4r}{4}$

$$\therefore\ x - \frac{p}{2} = \pm\frac{\sqrt{p^2 + 4r}}{2},$$

and $x = \frac{p + \sqrt{p^2 + 4r}}{2}$, or $\frac{p - \sqrt{p^2 + 4r}}{2}$, the former of which is poſitive, and the latter negative.

Ex. Let $x^2 - 3x = 10$

then $x^2 - 3x + \frac{9}{4} = \frac{49}{4}$;

$$\therefore\ x - \frac{3}{2} = \pm\frac{7}{2},$$

and $x = \pm\frac{7}{2} + \frac{3}{2} = 5$, or -2.*

THIRD FORM.

Let $x^2 - px = -r$; then $x^2 - px + \frac{p^2}{4} = \frac{p^2 - 4r}{4}$

$$\therefore\ x - \frac{p}{2} = \pm\frac{\sqrt{p^2 - 4r}}{2},$$

and $x = \frac{p + \sqrt{p^2 - 4r}}{2}$, or $\frac{p - \sqrt{p^2 - 4r}}{2}$, both impoſſible, or both poſitive, according as $4r$ is or is not greater than p^2.

---

* If -2 be ſubſtituted in the given equation, it becomes $-2\times-2 - 3\times-2 = 10$, or $4 + 6 = 10$.

If $p^2 = 4r$, $\sqrt{p^2 - 4r} = 0$; and $\frac{p + \sqrt{p^2 - 4r}}{2}$ and $\frac{p - \sqrt{p^2 - 4r}}{2} = \frac{p}{2}$.*

Ex. 1ſt. Let $x^2 - 9x = -14$.

Then $x^2 - 9x + \frac{81}{4} = \frac{81}{4} - 14 = \frac{25}{4}$

$\therefore x - \frac{9}{2} = \pm \frac{5}{2}$, and $x = \frac{9}{2} \pm \frac{5}{2} = 7$ or 2.

Ex. 2d. Let $x^2 - 8x = -16$.

Then $x^2 - 8x + 16 = -16 + 16 = 0$;

$\therefore x - 4 = 0$, and $x = 4$.

Ex. 3d. † Let $x^2 - 6x = -12$.

then $x^2 - 6x + 9 = -12 + 9 = -3$;

$\therefore x - 3 = \pm \sqrt{-3}$

and $x = 3 + \sqrt{-3}$, or $3 - \sqrt{-3}$.‡

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* In this caſe the equation may be ſaid to have two *equal* roots.

† This equation coincides with the equation $6x - x^2 = 12$, which is impoſſible; for by art. 245, $6x - x^2$ cannot be greater than 9, the ſquare of half 6.

‡ If $3 + \sqrt{-3}$ be ſubſtituted for x in the given equation, it becomes $\overline{3 + \sqrt{-3}}^2 - 6 \times \overline{3 + \sqrt{-3}} = -12$,

or $9 - 3 + 6\sqrt{-3} - 18 - 6\sqrt{-3} = -12$. and if $3 - \sqrt{-3}$ be ſubſtituted, it becomes $\overline{3 - \sqrt{-3}}\rceil^2 - 6 \times \overline{3 - \sqrt{-3}} = -12$,

or $9 - 3 - 6\sqrt{-3} - 18 + 6\sqrt{-3} = -12$.

FOURTH

FOURTH FORM.

Let $x^2 + px = -r$; then $x^2 + px + \frac{p^2}{4} = \frac{p^2 - 4r}{4}$

$$\therefore x + \frac{p}{2} = \pm \frac{\sqrt{p^2 - 4r}}{2}$$

and $x = \frac{\sqrt{p^2 - 4r} - p}{2}$, or $\frac{-\sqrt{p^2 - 4r} - p}{2}$, both impoſſible or both negative, according as $4r$ is or is not greater than p^2.

If $4r = p^2$, $\frac{\sqrt{p^2 - 4r} - p}{2} = \frac{-\sqrt{p^2 - 4r} - p}{2}$ $= -\frac{p}{2}$.

Ex. 1ſt. Let $x^2 + 9x = -14$.
then $x^2 + 9x + \frac{81}{4} = \frac{81}{4} - 14 = \frac{25}{4}$,
and $x + \frac{9}{2} = \pm \sqrt{\frac{25}{4}} = \pm \frac{5}{2}$;

$$\therefore x = \pm \frac{5}{2} - \frac{9}{2} = -2, \text{ or } -7.^*$$

Ex. 2d. Let $x^2 + 8x = -16$.
Then $x^2 + 8x + 16 = -16 + 16 = 0$;
and $x + 4 = 0$ $\therefore x = -4.$†

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* If -2 be ſubſtituted for x in the given equation, it becomes $-2 \times -2 + 9 \times -2 = -14$, or $4 - 18 = -14$, and if -7 be ſubſtituted, it becomes $-7 \times -7 + 9 \times -7 = -14$, or $49 - 63 = -14$.

† If -4 be ſubſtituted for x in the given equation, it becomes $-4 \times -4 + 8 \times -4 = -16$, or $16 - 32 = -16$.

Ex. 3d. Let $x^2 + 6x = -12$.

Then $x^2 + 6x + 9 = -12 + 9 = -3$,

and $x + 3 = \pm \sqrt{-3}$;

$\therefore x = -3 + \sqrt{-3}$, or $-3 + \sqrt{-3}$.*

248. It is obvious that any equation in which only two powers of the unknown quantity are found, and in which the index of one of thoſe powers is double that of the other, may be ſolved in the ſame manner as a quadratic equation.

[*] If $-3 \pm \sqrt{-3}$ be ſubſtituted for x in the given equation, it becomes $\overline{-3 \pm \sqrt{-3}}^2 + 6 \times \overline{-3 \pm \sqrt{-3}} = -12$, or $9 - 3 \mp 6\sqrt{-3} - 18 \pm 6\sqrt{-3} = -12$.

It may be uſeful to obſerve,

1ſt. That the negative root $\frac{-p - \sqrt{p^2 + 4r}}{2}$, of the firſt form, is, as to its magnitude, the ſame as the poſitive root of the ſecond form, and that if $-y$ be ſubſtituted for x in the firſt form, it will be converted into the ſecond form, for it will become $-y \times -y + p \times -y = r$, or $y^2 - py = r$.

2dly. That the negative root, $\frac{p - \sqrt{p^2 + 4r}}{2}$, of the ſecond form, is, as to its magnitude, the ſame as the poſitive root of the firſt form; and that if $-y$ be ſubſtituted for x in the ſecond form, it will be converted into the firſt form.

3dly. That the two (negative) roots of the fourth form are, as to their magnitude, the ſame as the two (poſitive) roots of the third form, and that if $-y$ be ſubſtituted for x in the fourth form, it will be converted into the third form.

For if $x^{2m} \pm px^m = \pm r$

$x^{2m} \pm px^m + \frac{p^2}{4} = \frac{p^2 \pm 4r}{4}$, and extracting the ſquare root on both ſides $x^m \pm \frac{p}{2} = \pm \frac{\sqrt{p^2 \pm 4r}}{2}$, whence $x^m = \frac{\pm \sqrt{p^2 \pm 4r} \mp p}{2}$

and $x = \overline{\frac{\pm\sqrt{p^2 \pm 4r} \mp p}{2}}\Big|^{\frac{1}{m}}$

If y be ſubſtituted for x^m, $x^{2m} = y^2$, and the equation becomes $y^2 \pm py = \pm r$, a quadratic, in which the values of y are the values of x^m.

EXAMPLES.

1ſt. Let $x^4 - 2x^2 = 8$.
Then $x^4 - 2x^2 + 1 = 9$, and $x^2 - 1 = \pm 3$.
Hence, $x^2 = \pm 3 + 1 = 4$, or -2,
and* $x = 2$, or -2, or $\sqrt{-2}$, or $-\sqrt{-2}$.

2d. Let $z - \frac{\sqrt{z}}{2} = 3$; then $z - \frac{\sqrt{z}}{2} + \frac{1}{16} = \frac{49}{16}$

hence $\sqrt{z} - \frac{1}{4} = \pm \frac{7}{4}$ and $\sqrt{z} = \pm \frac{7}{4} + \frac{1}{4}$

$= 2$ or $-\frac{3}{2}$;

$\therefore z = 4$ or $\frac{9}{4}$.

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* Although there be in this example four values of x, there are but two values of x^2, and x^2 is the root of the given equation conſidered as a quadratic, or as coinciding with $y^2 - 2y = 8$.

Ex. 3d. Let $y^6 + ry^3 + \frac{q^3}{27} = 0$.

Then $y^6 + ry^3 = -\frac{q^3}{27}$.

$$\therefore y^6 + ry^3 + \frac{r^2}{4} = \frac{r^2}{4} - \frac{q^3}{27}.$$

Hence, $y^3 + \frac{r}{2} = \pm\sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$

and $y^3 = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$

$$\therefore y = \sqrt[3]{-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}.$$

249. Examples of equations that may be reduced to the *quadratic form*.

Ex. 1st. Let $\frac{4x-3}{3x-7} - \frac{2x-3}{x-1} = 3$

$$4x - 3 - \frac{\overline{2x-3} \times \overline{3x-7}}{x-1} = 3 \times \overline{3x-7}$$

or $4x - 3 - \frac{6x^2 - 23x + 21}{x-1} = 9x - 21$

$$4x^2 - 7x + 3 - 6x^2 + 23x - 21 = 9x^2 - 30x + 21$$

$$11x^2 - 46x = -39$$

$$x^2 - \frac{46x}{11} = -\frac{39}{11}$$

$$x^2 - \frac{46x}{11} + \frac{\overline{23}|^2}{\overline{11}|^2} = -\frac{39}{11} + \frac{\overline{23}|^2}{\overline{11}|^2} = \frac{100}{\overline{11}|^2}.$$

$$x - \frac{23}{11} = \pm\frac{10}{11}$$

$$x = \frac{23}{11} \pm \frac{10}{11} = 3 \text{ or } \frac{13}{11}.$$

Ex. 2d. Let $x + \sqrt{ax + b} = c$

$$\sqrt{ax + b} = c - x$$

$$ax + b = \overline{c - x}^2 = c^2 - 2cx + x^2$$

$$x^2 - \overline{2c + a}\,.\,x = b - c^2$$

$$x^2 - \overline{2c + a}\,.\,x + \frac{\overline{2c+a}\rceil^2}{4} = b - c^2 + \frac{\overline{2c+a}\rceil^2}{4}$$

$$= \frac{4b + a^2 + 4ac.}{4}$$

$$x - \frac{2c + a}{2} = \pm \frac{\sqrt{4b + a^2 + 4ac}}{2}$$

$$x = \frac{2c + a}{2} \pm \frac{\sqrt{4b + a^2 + 4ac}}{2}$$

Theſe two values are the values of x in the equation $ax + b = c^2 - 2cx + x^2$, and it may happen that but *one* of them can be a value of x, in the equation $\sqrt{ax + b} = c - x$, provided that the expreſſion, $\sqrt{ax + b}$, be reſtricted to denote the poſitive root of $ax + b$. For the equation $ax + b = c^2 - 2cx + x^2$ may be derived either from the equation $\sqrt{ax + b} = c - x$ or from the equation $\sqrt{ax + b} = x - c$, which are two different ſuppoſitions.

It is obvious that if a, b and c repreſent poſitive quantities, $\frac{2c + a}{2} - \frac{\sqrt{4b + a^2 + 4ac}}{2}$ is the value of x that anſwers the condition of the equation $\sqrt{ax + b} = c - x$, becauſe the other root is greater than c, and by this hypotheſis x is leſs than c.

Sometimes, when by the obvious method of reduction, an equation would be obtained involving more than two powers of the unknown quantity, the quadratic form may be obtained by a deviation from the general rules.

Ex. 3d. Let $y^2 - 2y + 6\sqrt{y^2 - 2y + 5} = 11$. Adding 5 to each ſide $y^2 - 2y + 5 + 6\sqrt{y^2 - 2y + 5} = 16$, or $\overline{y^2 - 2y + 5}\rceil^{1} + 6 \times \overline{y^2 - 2y + 5}\rceil^{\frac{1}{2}} = 16$ an equation of the quadratic form; hence

$$y^2 - 2y + 5 + 6\sqrt{y^2 - 2y + 5} + 9 = 25$$

$$\sqrt{y^2 - 2y + 5} + 3 = \pm 5, \quad \sqrt{y^2 - 2y + 5} = 2 \text{ or } -8^*$$

$$\therefore y^2 - 2y + 5 = 4, \text{ or } 64$$

$$y^2 - 2y = -1 \text{ or } 59$$

$$y^2 - 2y + 1 = 0 \text{ or } 60$$

$$y - 1 = 0 \quad \text{or } \pm\sqrt{60}$$

$$y = 1 \quad \text{or } \pm\sqrt{60}$$

When there are more equations and unknown quantities than one, a ſingle equation involving only one of the unknown quantities muſt be obtained by the rules in the preceding chapter, or by other ſimilar methods; and if this equation be of the quadratic form, the values of the unknown quantities may be found.

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* If $\sqrt{y^2 - 2y + 5}$ be reſtricted to denote a poſitive quantity, it will have but one value;and $(\pm\sqrt{60})$ the two values of y derived from the equation $\sqrt{y^2 - 2y + 5} = -8$ will not anſwer the conditions of the original equation.

Ex. 4th. Let $\left\{\begin{matrix} x^2 + y^2 = 13 \\ x + y = 5 \end{matrix}\right\}$

From the 2d equ. $x = 5 - y$

$$\therefore \overline{5-y}^2 + y^2 = 13, \text{ or } 25 - 10y + 2y^2 = 13$$

$$y^2 - 5y = -6$$

$$y^2 - 5y + \frac{25}{4} = -6 + \frac{25}{4} = \frac{1}{4}$$

$$y - \frac{5}{2} = \pm \frac{1}{2}$$

$$y = \frac{5}{2} \pm \frac{1}{2} = 3, \text{ or } 2$$

$$x = 5 - y = 2, \text{ or } 3.$$

Ex. 5th. Let $\left\{\begin{matrix} x^2 + xy = 40 \\ 2xy - 3y^2 = 3 \end{matrix}\right\}$

From the 2d equ. $x = \frac{3+3y^2}{2y}$ and $x^2 = \frac{9+18y^2+9y^4}{4y^2}$

Hence, $\frac{9 + 18y^2 + 9y^4}{4y^2} + \frac{3+3y^2}{2} = 40$

$$9 + 18y^2 + 9y^4 + 6y^2 + 6y^4 = 160y^2$$

$$15y^4 - 136y^2 = -9$$

$$y^4 - \frac{136}{15}y^2 = -\frac{9}{15}$$

$$y^4 - \frac{136}{15}y^2 + \overline{\frac{68}{15}}\Big|^2 = -\frac{9}{15} + \overline{\frac{68}{15}}\Big|^2 = \frac{4489}{\overline{15}|^2}$$

$$y^2 - \frac{68}{15} = \pm \frac{\sqrt{4489}}{15} = \pm \frac{67}{15}$$

$$y^2 = \frac{68 \pm 67}{15} = 9, \text{ or } \frac{1}{15}$$

$$y = \pm 3, \text{ or } \pm \frac{1}{\sqrt{15}}. \quad x = \frac{3+3y^2}{2y} = \pm 5 \text{ or } \pm \frac{24}{\sqrt{15}}.$$

Ex. 6. Let $\left\{\begin{matrix} x^2 + y^2 = 73 \\ xy \quad = 24 \end{matrix}\right\}$

add. double the 2d equ. to the 1ſt $x^2+y^2+2xy=121$

ſubtracting . . . from . . $x^2+y^2-2xy=25$

$\therefore (\sqrt{x^2+y^2+2xy} =)\ x+y = \pm 11$

and $(\sqrt{x^2+y^2-2xy} =)\ x-y = \pm 5$

Hence by add. $2x = \pm 11 \pm 5$

$$x = \frac{\pm 11 \pm 5}{2} = 8 \text{ or } 3, \text{ or } -3 \text{ or } -8$$

$$y = \frac{24}{x} = 3 \text{ or } 8, \text{ or } -8 \text{ or } -3$$

Ex. 7. Let $\left\{\begin{matrix} \frac{x^2}{y} + \frac{y^2}{x} = a \\ x + y = b \end{matrix}\right\}$

From the 1ſt equ. $x^3 + y^3 = ayx$

From the 2d $x = b - y$ and $x^3 = \overline{b-y}|^3$

Hence, $b^3 - 3b^2y + 3by^2 = ay \times \overline{b-y} = aby - ay^2$

$\overline{3b+a} \times y^2 - \overline{3b^2+ab} \times y = -b^3$

$$y^2 - by = -\frac{b^3}{3b+a}$$

$$y^2 - by + \frac{b^2}{4} = -\frac{b^3}{3b+a} + \frac{b^2}{4} = \frac{b^2a - b^3}{12b+4a}$$

$$y - \frac{b}{2} = \pm\sqrt{\frac{b^2a - b^3}{12b+4a}}$$

$$\therefore y = \frac{b}{2} \pm \sqrt{\frac{b^2a - b^3}{12b+4a}}$$

and $x = b - y = \dfrac{b}{2} \mp \sqrt{\dfrac{b^2a - b^3}{12b+4a}}$.

Sometimes inſtead of deducing an equation involving one of the unknown quantities that are contained in the given equations, it will be neceſſary (in order that the form may be quadratic) to deduce an equation involving ſome other quantity, which is dependant on the unknown quantities, and from whoſe values their value may be obtained.

Ex. 8th. Let $\left\{\begin{matrix} x + y = 4 \\ x^5 + y^5 = 244 \end{matrix}\right\}$

put $x = 2 + v$

then $y = 2 - v$

Hence, $\overline{2 + v}\rceil^5 + \overline{2 - v}\rceil^5 = 244$

$$\left.\begin{matrix} \text{or } 32 + 80v + 80v^2 + 40v^3 + 10v^4 + v^5 \\ + 32 - 80v + 80v^2 - 40v^3 + 10v^4 - v^5 \end{matrix}\right\} = 244$$

i. e. $64 * + 160v^2 * + 20v^4 * = 244$

Hence, $v^4 + 8v^2 = 9$

$v^4 + 8v^2 + 16 = 9 + 16 = 25$

$v^2 + 4 = \pm 5$

$v^2 = \pm 5 - 4 = 1 \text{ or } - 9.$

$v = \pm 1 \text{ or } \pm \sqrt{-9}$

$x = 2 + v = 3 \text{ or } 1, \text{ or } 2 + \sqrt{-9} \text{ or } 2 - \sqrt{-9}$

$y = 2 - v = 1 \text{ or } 3, \text{ or } 2 - \sqrt{-9} \text{ or } 2 + \sqrt{-9}$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

Problem 1ſt. What two numbers are thoſe, whoſe difference is 15, and half of whoſe product is equal to the cube of the leſs?

Let x denote the leſs of the two numbers; then $x+15$ will denote the greater; and by the queſtion $\frac{x^2+15x}{2}=x^3$. Hence $x+15=2x^2$

$$x^2-\frac{x}{2}=\frac{15}{2} \text{ and } x^2-\frac{x}{2}+\frac{1}{16}=\frac{15}{2}+\frac{1}{16}=\frac{121}{16}.$$

$$x-\frac{1}{4}=\pm\frac{11}{4}$$

$$x=\frac{1}{4}\pm\frac{11}{4}=3 \text{ or } -\frac{5}{2}.$$

The numbers required therefore are 3 and (3 + 15 =) 18.

Problem 2d. A farmer bought a lot of oxen for eighty guineas; if he had had four more for the ſame ſum they would have coſt him a guinea a piece leſs. Required the number of oxen, and the price of each?

Let x denote the number of oxen; then ſince x oxen coſt eighty guineas, one ox coſt an xth part of eighty guineas, or $\frac{80}{x}$ guineas; and if he had had four more for the ſame ſum, the price of each would have been $\frac{80}{x+4}$ guineas; but by the queſtion this latter price is one guinea leſs than the former.

$$\therefore \frac{80}{x+4} = \frac{80}{x} - 1$$

$$\text{Hence, } 80 = \frac{80x + 320}{x} - x - 4$$

$$80x = 80x + 320 - x^2 - 4x$$

$$x^2 + 4x = 320;\ x^2 + 4x + 4 = 324$$

$$x + 2 = \pm 18$$

$$x = \pm 18 - 2 = 16 \text{ or } - 20$$

$$\text{and } \frac{80}{x} = 5 \text{ or } - 4.$$

Therefore the number of oxen was 16, and the price of each 5 guineas.

Problem 3d. Two travellers, *A* and *B* ſet out from two places *C* and *D* at the ſame time, *A* from *C* bound for *D*, and *B* from *D* bound for *C*; when they met and had computed their travels, it was found that *A* had travelled thirty miles more than *B*, and that, at their rate of travelling, *A* expected to reach *D* in four days, and *B* to reach *C* in nine days. Required the diſtance between the two places *C* and *D*?

Let x denote the number of miles between *C* and *D;* then it is evident that *A* and *B* both together had travelled x miles when they met; therefore as much as the miles travelled by *A* exceeded $\frac{x}{2}$, juſt ſo much did the miles travelled by *B* fall ſhort of $\frac{x}{2}$; but, by the ſuppoſition, *A*'s miles exceeded thoſe of *B* by 30; therefore *A* muſt have travelled $\frac{x}{2} + 15$ or $\frac{x+30}{2}$ miles; and *B* muſt have travelled $\frac{x}{2} - 15$ or $\frac{x-30}{2}$ miles; therefore the remaining part of *A*'s journey is $\frac{x-30}{2}$ miles, which he expects to perform in four days; and the remaining part of *B*'s journey is $\frac{x+30}{2}$ miles, which he expects to perform in nine days. Now, if *A* can travel $\frac{x-30}{2}$ miles in four days, he will travel one mile in an $\frac{x-30}{2}$ part of four days; i. e. in $4 \div \frac{x-30}{2}$ days, and he will travel $\frac{x+30}{2}$ miles in $\frac{x+30}{2} \times 4 \div \frac{x-30}{2}$ days, i. e. in $\frac{\overline{x+30}\times 4}{x-30}$ days; and by ſimilar reaſoning *B* can travel $\frac{x-30}{2}$ miles in $\frac{\overline{x-30}\times 9}{x+30}$ days. Therefore from the time of their ſetting out to the time of their meeting *A*, has travelled $\frac{\overline{x+30}\times 4}{x-30}$ days, and *B* has travelled $\frac{\overline{x-30}\times 9}{x+30}$ days; but they ſet out at the ſame time; hence,

$$\frac{\overline{x+30}\times 4}{x-30} = \frac{\overline{x-30}\times 9}{x+30}$$

$\overline{x+30}\rceil^2\times 4 = \overline{x-30}\rceil^2\times 9$ ∴ extracting the ſquare roots on both ſides $\overline{x+30}\times 2 = \pm\overline{x-30}\times 3$

Hence, $2x+60 = 3x-90$, or $-3x+90$ whence $x = 150$, or 6. Therefore the diſtance between the two places *C* and *D* is 150 miles.

Problem 4th. Two travellers, *A* and *B*, ſet out from two places *C* and *D* at the ſame time; *A* from *C* with a deſign to paſs through *D*, and *B* from *D* with a deſign to travel the ſame way; after *A* had overtaken *B*, and they had computed their travels, it was found that the diſtance *A* had travelled together with the diſtance *B* had travelled made up 30 miles; that *A* had paſſed through *D* four hours before, and that *B*, at his rate of travelling, was a nine day's journey diſtant from *C*. Required the diſtance between the two places *C* and *D*?

Let x denote the number of miles between C and D; then it is evident that A muſt have travelled x miles more than B; but they both together had travelled 30 miles by the ſuppoſition; therefore A muſt have travelled $15+\frac{x}{2}$ or $\frac{30+x}{2}$ miles, and B muſt have travelled $15-\frac{x}{2}$ or $\frac{30-x}{2}$ miles; therefore A's diſtance from D, after he had overtaken B, was $\frac{30-x}{2}$ miles, which he had travelled in four days; and B's diſtance from C was $\frac{30+x}{2}$ miles, which he could travel in nine days. Hence, by reaſoning ſimilar to that in the laſt problem, it appears, that A was $\frac{\overline{30+x}\times4}{30-x}$ days in travelling his $\frac{30+x}{2}$ miles; and that B was $\frac{\overline{30-x}\times9}{30+x}$ days in travelling his $\frac{30-x}{2}$ miles; but as they both ſet out at the ſame time, theſe times are equal; therefore $\frac{\overline{30+x}\times4}{30-x}=\frac{\overline{30-x}\times9}{30+x}$ and

$$\overline{30+x}|^2\times4=\overline{30-x}|^2\times9$$

$$\therefore\ \overline{30+x}\times2=\pm\,\overline{30-x}\times3.$$

Hence, $2x+60=90-3x$ or $3x-90$; whence $x=6$ or 150.

Therefore the diſtance between the two places C and D is 6 miles.

Problem 5th. In a parcel conſiſting of twenty gold and ſilver coins, each gold coin was worth as many ſhillings as there were ſilver coins; and each ſilver coin was worth as many ſhillings as there were gold coins, and the parcel was worth 150 ſhillings. Required the number of each?

Let x denote the number of gold coins, and y the number of ſilver coins; then ſince each gold coin is worth y ſhillings, the x gold coins are worth xy ſhillings; and ſince each ſilver coin is worth x ſhillings, the y ſilver coins are worth yx ſhillings; therefore the worth of the whole parcel is $2xy$ ſhillings.

Hence, by the queſtion $\left\{\begin{matrix} x + y = 20 \\ 2xy = 150 \end{matrix}\right\}$

$\therefore\ x = 20 - y$ and $y \times \overline{20 - y} = 75$

$$\begin{aligned} \text{Hence, } y^2 - 20y &= -75 \\ y^2 - 20y + 100 &= 25 \\ y - 10 &= \pm\sqrt{25} = \pm 5 \\ y &= 10 \pm 5 = 15 \text{ or } 5 \\ x\ (= 20 - y) &= 5 \text{ or } 15 \end{aligned}$$

The parcel therefore conſiſted either of 5 gold coins worth 15 ſhillings a piece, and 15 ſilver coins worth 5 ſhillings a piece, or of 15 gold coins worth 5 ſhillings a piece, and 5 ſilver coins worth 15 ſhillings a piece.

Problem 6th. A tradeſman having loſt 200l. found that if each pound that remained were converted into his original fortune, he ſhould be worth 1500l. What was he worth?

Let x denote the number of pounds he was worth at firſt; then $\overline{x - 200}$ l. is his worth after the loſs of 200l. and if for each pound in theſe $\overline{x - 200}$ l. xl. be ſubſtituted, the amount will be $x \times \overline{x - 200}$ l.

∴ by the queſtion $x^2 - 200x = 1500$
which equation reſolved gives give $x = 500$ or -300
therefore his original fortune was 500l.

SCHOLIUM.

250. It is evident, that when all the conditions of a problem are expreſſed in the equation produced by it, the number of poſitive roots to the equation can neither be greater nor leſs than the number of anſwers the problem will admit of; and this, whether the equation produced by the problem be a quadratic equation, or an equation of any higher dimenſion whatſoever. For what is ſuch an equation but a propoſition expreſſing in ſymbols, or the language of algebra, the very ſame ideas, ſuppoſitions and inquiries, that are

expreſſed in the problem in words or common language? Any queſtion or problem therefore which produces an equation either of the form $x^2 + px = r$, or of the form $x^2 - px = r$, will admit of only one anſwer. But any qneſtion or problem, which produces an equation of the form $px - x^2 = r$, will admit of two anſwers, unleſs there be ſome condition in the queſtion or problem, which cannot be expreſſed in the equation, and which determines it to have but one anſwer. Thus, in the 2d of the foregoing problems the equation has but one poſitive root, and the problem but one anſwer: But in the 5th problem, the equation has two poſitive roots, and the problem two anſwers. If the number of gold coins had been reſtricted to have been leſs than the number of ſilver coins, the problem would have had but one anſwer, and yet the equation would have had two poſitive roots, for the method of ſolution would have been the ſame: And in this caſe there would have been a condition in the problem, which could not have been expreſſed in the equations.*

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* If it be left indifferent whether x or y be the greater of the two unequal parts, into which 20 is divided, we muſt take the root of the equation, $y^2 - 20y + 100 = 25$, to be both $y - 10 = 5$, and $10 - y = 5$, or (which is the ſame thing) $y - 10 = \pm 5$. If y be determined to denote the greater, we muſt take the root of the equation to be $y - 10 = 5$, becanſe then y will be leſs than 10: and if y be determined to denote the leſs, we muſt take the root of the equation to be $10 - y = 5$.

If, when one of the values of the unknown quantity in the final equation is negative, a ſymbol with the negative ſign prefixed be ſubſtituted in the equations derived from the problem, for the ſymbol contained in that final equation, and in the ſame manner other ſymbols with the negative ſign prefixed be ſubſtituted for each of the other ſymbols, whoſe values, correſponding with the negative value in the ſaid final equation, are negative, theſe equations will be converted into others, in which the poſitive values of the ſymbols, will be the ſame,* as to magnitude, as the negative values of the ſymbols in the former equations, and *vice verſa;* and theſe poſitive values will be the anſwer to that problem, whoſe conditions are expreſſed by the transformed equations.

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* For let certain equations involving the ſymbols z, y, &c. be thus transformed by ſubſtituting $-z$ for z; $-y$ for y; &c. then, whatever would reſult from ſubſtituting in the former equations known poſitive quantities, $+a$, $+b$, &c. for z, y, &c. the ſame muſt reſult from ſubſtituting in the transformed equations $-a$, $-b$, &c. for z, y, &c. And whatever would reſult from ſubſtituting in the former equations for z, y, &c. $-a$, $-b$, &c. the ſame muſt reſult from ſubſtituting in the transformed equations for z, y, &c. $+a$, $+b$, &c. Therefore whatever poſitive quantities would juſtify the former equations, the ſame quantities taken negatively would juſtify the transformed equations; and *vice verſa*; or the poſitive values of z, y, &c. in the former equations are the negative values of z, y, &c. in the transformed equations. For example, becauſe in the equations $x+y+xy=5$, and $x-y=1$, the poſitive values of x and y are 2 and 1 reſpectively, and the negative values -3 and -4 reſpectively, it follows that in the equations, which reſult from ſubſtituting $-x$ for x, and $-y$ for y, viz. the equations $xy-x-y=5$, and $y-x=1$, the poſitive values of x and y are 3 and 4, and the negative values -2 and -1 reſpectively.

Hence it appears that the negative root of a quadratic equation is (as to its magnitude) the anſwer to ſome problem *ſimilar*, (on account of the ſimilarity of the equations before and after tranſformation) to that, from which the equation was derived. The two problems will generally convert one into the other, by changing one or more words into their contraries, as gain into loſs; progreſs into regreſs; increaſe into decreaſe; and the like. Thus, in the equation to the 1ſt problem, if $-x$ be ſubſtituted for x, it becomes

$$\frac{-x \times \overline{15-x}}{2} = -x^3 \quad \text{or}$$

$$\frac{x \times \overline{15-x}}{2} = x^3,$$ where the poſitive value of x is $\frac{5}{2}$, but the equation $\frac{x \times \overline{15-x}}{2} = x^3$ is a ſtatement of the conditions of the following problem:

"*What two numbers are thoſe, whoſe* ſum *is* 15, *and half whoſe product is equal to the cube of the leſs.*"

leſs."* Which problem is therefore anſwered by the negative value of x in the 1ſt problem.

In the equation to the 2d problem if $-x$ be ſubſtituted for x, it becomes $\frac{80}{-x+4} = \frac{80}{-x} - 1$; or $\frac{80}{x-4} = \frac{80}{x} + 1$, where the poſitive value of x is 20. But this equation is a ſtatement of the conditions of the following problem. *A farmer* ſold *a certain number of oxen for* 80 *guineas; had he ſold four* leſs *for the ſame ſum, he would have* received *a guinea a piece* more *for them. Required the number of oxen?* Which problem is therefore anſwered by the negative value of x in the 2d problem.

In the equation to the 3d problem $x+30$ is the expreſſion, for which two values are found, (one of a poſitive form, and the other of a negative form) from the reſolution of the quadratic equation;

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* x, in the above equation, will neceſſarily repreſent the *leſs* of the two numbers; whoſe ſum is 15, becauſe otherwiſe $x \times \overline{15-x}$ would be leſs than x^2, and $\frac{x \times \overline{15-x}}{2}$ leſs than $\frac{x^2}{2}$, conſequently x^2 would be leſs than $\frac{x^2}{2}$, and x leſs than $\frac{1}{2}$; or the greater of two numbers, whoſe ſum is 15, leſs than $\frac{1}{2}$; which is abſurd.

tion; ſubſtituting therefore $-x-30$ for $x+30$ the equation becomes

$$\frac{-\ \overline{x+30}}{x-30}\times 4=\frac{\overline{x-30}\times 9}{-x+30}\text{ or }\frac{\overline{x+30}\times 4}{30-x}=\frac{\overline{30-x}\,.\,9}{x+30}$$

in which that value of x, which would be derived from the poſitive root of the reſulting quadratic, $4\times\overline{x+30}\rceil^2=9\times\overline{30-x}\rceil^2$, is 6. But the above equation is a ſtatement of the 4th problem, which is therefore anſwered by that value of x, which is derived from the negative root of the quadratic in the 3d problem; i. e. from the equation $2\times\overline{x+30}=-3\times\overline{x-30}$.

In a ſimilar manner it may be ſhewn that the value of x, obtained from the negative root of the quadratic in the 4th problem, i. e. from the equation $2\times\overline{x+30}=-3\times\overline{30-x}$, *muſt neceſſarily* be the anſwer to the 3d problem.

In the equation to the 6th problem, if $-x$ be ſubſtituted for x, it becomes $x^2+200x=1500$ where the poſitive value of x is 300; but this equation is a ſtatement of the following problem:

" *A tradeſman having* gained 200*l.* *found that if for each pound of* his preſent worth *there were ſubſtituted his original fortune, he ſhould be worth* 1500*l.*

1500*l.* *Required his original fortune?"*——Or of the * following problem:

" *A tradeſman having loſt* 200*l.* *found, that if for each pound of his debts, there were ſubſtituted his debts previous to the loſs, they would amount to* 1500*l.* *Required the amount of his debts?"* Which two problems are therefore anſwered by the negative value of x in the 6th problem.

When the roots of an equation are found to be impoſſible, no ſuch transformation of the problem can give them any ſignification.—In this caſe the conditions of the problem are inconſiſtent with each other, or ſuch as never either *did*, or *could* take place. Thus, if in the 5th problem 270 were ſubſtituted inſtead of 170, the roots of the equation would be found impoſſible, and the problem, if given as a circumſtance which really did take place, would be falſe or miſtated, and if given as a fictitious caſe would be abſurd; for 20 cannot be divided into two parts, the double of whoſe product is greater than the double of 10 × 10.

In this and the like inſtances, the abſurdity may be diſcovered, either by conſidering the neceſſary limitations of the conditions of the problem, or by reſolving the equation; for the final equation will always involve ſome impoſſible expreſſion.

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* There is likewiſe another problem (ſimilar to problem 6th) of which the equation $x^2-200x=1500$ is a ſtatement. And the ſame thing may be obſerved of moſt equations.

CHAPTER IX.

OF RATIOS.

251. RATIO is the relation which one quantity has to another in reſpect of magnitude. The idea ariſes from conſidering what multiple, part or parts the one magnitude is of the other. When one magnitude is known to be the ſame multiple, part or parts of a ſecond magnitude, that a third is of a fourth, there is evidently a certain relation between the firſt and ſecond magnitudes, which obtains alſo between the third and fourth; this relation is called *Ratio*, and in common language, "*Proportion*,"—there being ſaid to be the ſame proportion between the third and fourth as between the firſt and ſecond. If, on the contrary, the third be not the ſame multiple, part or parts of the fourth that the firſt is of the ſecond, then the relation we are ſpeaking of is not the ſame between between the third and fourth, as between the firſt and ſecond. Thus, if any two magnitudes be denoted by *A* and *B*, the ratio of *A* to half *A* coincides with the ratio of *B* to half *B*; but the ratio of *A* to half *A* is different from the ratio of *B* to a third part of *B*.

Hence it immediately appears, that equality between the differences of the terms in different ratios is not the characteriſtic of equality in the ratios; for the difference between A and half A is half A; and the difference between B and half B, is half B; but between half A and half B there may ſubſiſt any degree of inequality whatſoever; indeed they cannot be equal, if A and B denote two different magnitudes; and on the other hand, although the difference between A and $A + C$ is the ſame as the difference between B and $B + C$, yet the ratio of A to $A + C$ is different from the ratio of B to $B + C$.

Thus, the ratio of 2 to 1 is the ſame as the ratio of 10 to 5, becauſe 2 is the ſame multiple of 1 that 10 is of 5, viz. the double of it; yet the difference between 2 and 1 is much leſs than the difference between 10 and 5. On the other hand the ratio of 9 to 10 is different from the ratio of 99 to 100, although the difference between 9 and 10 is the ſame as the difference between 99 and 100, for 9 is but 9 10ths of 10, whereas 99 is a much greater part of 100.*

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* The idea of ratio is eaſily diſcriminated from all others, and is indeed common in the minds of moſt men, being very frequently excited by the occurrences of life; and were it otherwiſe, no mathematical demonſtration would be in the leaſt invalidated, for "equality of ratio," or "ſameneſs of ratio," is, after all, only a phraſe, uſed to avoid prolixity, denoting a certain hypotheſis——viz. that of the

252. *Def.* If of four quantities the firſt be the ſame multiple, part or parts of the ſecond that the third is of the 4th, the ratio of the firſt to the ſecond is ſaid to be the ſame as, or equal to, the ratio of the third to the fourth.

253. Ratios are generally expreſſed by two points placed between the terms. Thus the ratio of a to b is expreſſed by $a : b$. The former term (a) is called the antecedent, and the latter term (b) is called the conſequent. The ſimilarity or equality of two ratios is expreſſed by four points placed between the ratios. Thus, if the ratio $a : b$ be the ſame as the ratio $c : d$ it is expreſſed by $a : b :: c : d$, which is read thus: a is to b as c to d.

254. Two equal ratios conſtitute what is called a proportion, and the four terms are ſaid to be proportionals. In the propoſition, the firſt and laſt

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four quantities under contemplation, the firſt and third are the ſame multiple part or parts of the ſecond and fourth reſpectively: and the following demonſtrations only ſhew, in what inſtances this condition obtains.

It may be further obſerved, that as the propoſitions do not at all depend on any abſtract idea annexed to the word "ratio," they might all of them be both enunciated and demonſtrated without uſing that or any other word of ſimilar import. For example, the article might be enunciated thus:

"If a be the ſame multiple, part or parts of b that c is of d, a will be the ſame multiple part or parts of c that b is of d."

laſt terms are called the extremes, and the ſecond and third, the means.

255. A proportion *may* conſiſt of but three *different* terms; for the conſequent of the firſt ratio may be the antecedent of the ſecond ratio.

256. If four quantities be proportionals, the quotient of the firſt divided by the ſecond, is equal to the quotient of the third divided by the fourth. For, if $a:b::c:d$, then ſince $a=\frac{a}{b}\times b$, or a is a bth parts of b, c is a bth parts of d by the definition, or $c=\frac{a}{b}\times d$

$\therefore$ dividing by d, $\frac{a}{b}=\frac{c}{d}$.[*]

257. If, of four quantities, the quotient of the firſt divided by the ſecond, be equal to the quotient of the third divided by the fourth, the quantities are proportionals.

For if $\frac{a}{b}=\frac{c}{d}$, $c=\frac{a}{b}\times d$; but $a=\frac{a}{b}\times b$.

$\therefore$ (by the definition) $a:b::c:d$.

258. In a proportion the product of the extremes is equal to the product of the means.

For $a=\frac{a}{b}\times b$. $\therefore$ if the four terms be proportional, $c = \frac{a}{b} \times d$, and (multiplying by b) $ad = bc$.

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[*] The equality of theſe quotients is, by many writers, made the criterion of proportionality.

259. Cor. If the two mean terms be equal, the product of the extremes is equal to the ſquare of the mean term.

260. Any three terms in a proportion being given, the fourth may be determined from the equation, *product of extremes* = *product of means.*

If there be but two *different* terms in the proportion, any two being given, the third may be determined from the ſame equation.

261. If two products be equal, the four factors form a proportion, when placed in any order ſo that the one product forms the mean terms, and the other the extreme terms.

For let $ad = bc$.

Then $a = \frac{c}{d} \times b$; but $c = \frac{c}{d} \times d$ $\therefore a : b :: c : d$.

Again, $b = \frac{d}{c} \times a$; but $d = \frac{d}{c} \times c$ $\therefore b : a : d : c$

Again, $a = \frac{b}{d} \times c$; but $b = \frac{b}{d} \times d$ $\therefore a : c :: b : d$

and in the ſame manner it may be ſhewn of the remaining orders:

262. Cor. 1ſt. If four terms be proportional, they are proportional alſo when taken inverſely, or the ſecond is to the firſt as the fourth to the third.

263. Cor. 2d. If four terms be proportionals, they are proportionals alſo when taken *alternately*, or the firſt is to the third as the ſecond to the fourth.

264. Ratios that are equal to the ſame ratio, are equal to one another.

For if $a : b :: c : d$ and $c : d :: e : f$, then becauſe a is the ſame part, multiple or parts of b that c is of d, and whatever part, multiple or parts c is of d, the ſame is e of f, a muſt be the ſame part, multiple or parts of b that e is of f, or $a : b :: e : f$.

265. If the terms of a ratio be multiplied by the ſame quantity, the ratio will not be altered.

For let the ratio be $a : b$, and let m denote any number whole or fractional. Then whatever part, multiple or parts a is of b, the ſame muſt ma be of mb.

$\therefore a : b :: ma : mb$.

266. If four terms be proportionals, they will alſo be proportionals in the following caſes:

1ſt. If the firſt and ſecond terms be multiplied or divided by the ſame quantity. 2dly. If all the terms be multiplied or divided by the ſame quantity. 3dly. If the firſt and ſecond be multiplied or divided by any quantity, and the third and fourth by any other quantity. 4thly. If the firſt and third be multiplied or divided by any quantity. 5thly. If the firſt and third be multiplied or divided by any quantity, and the ſecond and fourth by any other quantity.

For let $a : b :: c : d$, and let m and n repreſent any two numbers whatſoever whole or fractional. Then in the ſame manner as the two preceding articles were demonſtrated, it may be ſhewn, that

$ma : mb :: c : d$; $ma : mb :: mc : md$;
$ma : mb :: nc : nd$; $ma : b :: mc : d$
and $ma : nb :: mc : nd$.

The ſame concluſion may alſo be readily deduced from art. 261, for in all theſe caſes the product of the extremes is equal to the product of the means.

267. If four quantities be proportionals, their reciprocals are alſo proportionals.

For $\frac{1}{a} = \frac{b}{a} \times \frac{1}{b}$, and $\frac{1}{c} = \frac{d}{c} \times \frac{1}{d} =$ (if $a:b::c:d$) $\frac{b}{a} \times \frac{1}{d}$. $\therefore$ (by definition) $\frac{1}{a} : \frac{1}{b} :: \frac{1}{c} : \frac{1}{d}$.

268. If four quantities be proportionals, their like powers, or roots, or powers of roots will alſo be proportionals. For let $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, and $\frac{a^m}{b^m} = \frac{c^m}{d^m}$ $\therefore a^m : b^m :: c^m : d^m$, where m may denote any number whole or fractional.

269. If four terms be proportionals, the ſum of the firſt and ſecond will be to the ſecond or firſt as the ſum of the third and fourth to the

fourth or third. For let $a:b::c:d$, then $a+b=\overline{1+\frac{a}{b}}\times b$; and $c+d=\overline{1+\frac{c}{d}}\times d$ $=\overline{1+\frac{a}{b}}\times d$ ∴ (by definition) $a+b:b::c+d:d$.

Again, $a+b=\overline{1+\frac{b}{a}}\times a$; and $c+d=\overline{1+\frac{b}{a}}\times c$.

$\therefore a+b:a::c+d:c$,

270. If four terms be proportionals, the difference of the firſt and ſecond is to the ſecond or firſt, as the difference of the third and fourth to the fourth or third. For let $a:b::c:d$; then $a\sim b=\overline{1\sim\frac{a}{b}}\times b$, and $c\sim d=\overline{1\sim\frac{c}{d}}\times d=\overline{1\sim\frac{a}{b}}\times d$

∴ by definition $a\sim b:b::c\sim d:d$

and in the ſame manner it appears, that $a\sim b:a::c\sim d:c$,

271. Cor. to the two laſt articles. If $a:b::c:d$ $a+b:a\sim b::c+d:c\sim d$.

272. If there be any number of equal ratios, as one antecedent is to its conſequent, ſo is the ſum of all the antecedents to the ſum of all the conſequents.

For let the equal ratios be $a:b$, $c:d$, $e:f$, &c. Then $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$ &c. hence $a+c+e+$ &c, $\left(=b\times\frac{a}{b}+d\times\frac{c}{d}+f\times\frac{e}{f}+\text{\&c.}\right)=$

$\frac{a}{b} \times \overline{b + d + f + \&c.}$; alſo $a = \frac{a}{b} \times b$

$\therefore$ by def. $a : b :: a + c + e + \&c. : b + d + f + \&c.$

273. If the correſponding terms of two or more proportions be multiplied together, the reſulting terms will be proportionals.

For, let $a : b :: c : d$, and $e : f :: g : h$,

then becauſe $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$ $\quad \frac{ae}{bd} = \frac{cg}{dh}$

$\therefore ae : bf :: cg : dh$, and it is evident that the ſame method may be extended to any number of proportions.

274. If there be two ſets of quantities, and the ratio of the firſt to the ſecond in one ſet be the ſame as the ratio of the firſt to the ſecond in the other ſet, and the ratio of the ſecond to the third in the former ſet be the ſame as the ratio of the ſecond to the third in the latter ſet, and ſo on throughout, the ratio of the firſt to the laſt in one ſet ſhall equal the ratio of the firſt to the laſt in the other ſet.

For let the firſt ſet conſiſt of the terms $a, b, c,$ &c, and the ſecond ſet of the terms $m, n, p,$ &c.

And let $a : b :: m : n$

And $b : c :: n : p$

then $ab : bc :: mn : np$ (art. 273)

$\therefore$ (Art. 266,) $a : c :: m : p$

Again, let $c : d :: p : q$
then $ac : cd :: mp : pq$
$\therefore a : c :: m : q$
And ſo on for any other number of terms.

275. If there be a ſet of ratios $a : b$, $c : d$, $e : f$, &c, equal or unequal, and another ſet $m : n$, $n : p$, $p : q$, &c. reſpectively equal to the former; and in which the conſequent of the firſt ratio is the antecedent of the ſecond, and the conſequent of the ſecond the antecedent of the third, and ſo on; then whatever be the number of ratios, the ratio of the product of all the antecedents in the firſt ſet to the product of all the conſequents, is the ſame as the ratio of the firſt term to the laſt in the other ſet.

For let $a : b :: m : n$
And $c : d :: n : p$

then $ac : bd :: (mn : np)\ m : p$
Again, let $e : f :: p : q$
Then $ace : bdf\ (:: mp : pq)\ m : q$. &c.

Obſ. m and n may be taken equal to a and b reſpectively; for they repreſent any quantities whoſe ratio is the ſame as the ratio of a to b.

276. Cor. When any number of terms are continued proportionals, the ratio of the firſt term to the third is equal to the ratio of the ſquare of the firſt to the ſquare of the ſecond; and the ratio of

the firſt to the fourth is equal to the ratio of the cube of the firſt to the cube of the ſecond; and ſo on.

$$\begin{array}{ll}
\text{For if} & a : b :: b : c :: c : d :: d : e :: \&c. \\
\text{Then ſince} & a : b :: a : b \\
\text{And} & a : b :: b : c \\
\hline
 & a^2 : b^2 \; (:: ab : bc) :: a : c \\
\text{And} & a : b \; :: \qquad c : d \\
\hline
\therefore & a^3 : b^3 \; (:: ac : cd) :: a : d \\
 & \&c.
\end{array}$$

277. *Def.* Of any number of terms placed in any order, the firſt is ſaid to have to the laſt a ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and ſo on to the laſt term. Thus, any ratio $a : m$ is ſaid to be compounded of the ratios $a : b$, and $b : m$, or of the ratios $a : c$, $c : b$ and $b : m$, or of any other ratios whatſoever, provided the firſt antecedent be a, each other antecedent be the preceding conſequent, and the laſt conſequent be m.

278. *Obſ.* Properties relative to the intermediate ratios being found to attach to this ratio of the extreme terms, the name "*Compound ratio*" is given, to avoid circumlocution in the enunciation and demonſtration of propoſitions.

For example, art. 274, by means of this name is more briefly enunciated thus: "Ratios compounded of the ſame ratios are equal."

Which might at firſt ſight ſeem ſelf-evident, as if by it were meant, that equal things put together formed equal reſults; but " compound" is only a technical word expreſſive of a certain hypotheſis, as explained in the above definition.

See SIMSON's EUCLID, Note in Prop. 23, B. 6.

279. *Def.* In a ſeries of continued proportionals, the firſt term is ſaid to have to the third the duplicate ratio of that which it has to the ſecond; and to the fourth the triplicate ratio of that which it has to the ſecond; and ſo on. And converſely, the firſt term is ſaid to have to the ſecond the ſubduplicate ratio of that which it has to the third; and the ſubtriplicate ratio of that which it has to the fourth; and ſo on.

Thus, if $a : b :: b : c :: c : d ::$ &c.

$a : c$ is the duplicate ratio of $a : b$

$a : d$ is the triplicate ratio of $a : b$

and converſely $a : b$ is the ſubduplicate ratio of $a : c$, and the ſubtriplicate ratio of $a : d$.

Obſ. Duplicate ratios, triplicate ratios, &c. are particular caſes of compound ratio, viz. that caſe where the ratios compounded are equal among themſelves. For the ratio $a : c$ is compounded of the ratio $a : b$ and $b : c$; and the ratio $a : d$ is compounded of the ratios $a : b$, $b : c$ and $c : d$. See art. 277.

280. Cor. The duplicate ratio of $a : b$ is equal to the ratio $a^2 : b^2$; the triplicate ratio of $a : b$ is equal

to the ratio $a^3 : b^3$; &c. And converſely the ſubduplicate ratio of $a : b$ is equal to the ratio $a^{\frac{1}{2}} : b^{\frac{1}{2}}$; and the ſubtriplicate ratio of $a : b$ is equal to the ratio $a^{\frac{1}{3}} : b^{\frac{1}{3}}$; &c. See art. 276.

OF RATIOS CONSIDERED AS QUANTITY.

281. Every quality, property or relation, which having a general name, excites differently modified ideas in different caſes, is ſubject to menſuration, by means of an arbitrary definition.*

Among

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* Whatever excites an idea, or has a name, may be ſuppoſed repeated, and the number and law of the repetitions be made the ſubject of mathematical inquiry; but excepting lines, ſurfaces and ſolids, nothing which preſents itſelf to the mind as *one and entire*, is, in its own nature, diviſible into parts homogeneous to the whole, and to one another: therefore no two things of the ſame denomination, except two lines, or two ſurfaces, or two ſolids, can, by their own nature, and without a definition, be compared together with reſpect to magnitude. For inſtance, if there be two ſubſtances having the property called ſmoothneſs, but ſo as to convey a different ſenſation; the one ſmoothneſs is (by analogy) ſaid to be greater than the other, but how much the greater ſmoothneſs exceeds the leſs, cannot be determined, except by means of ſome artificial definition. One ſmoothneſs does not preſent itſelf to the mind as double or triple, &c. of any other, but only as different.

Again, If there be two different heats, the more powerful of the two is (by analogy) called greater than the other, but how much the greater exceeds the leſs, is not cognizable from the nature of the thing; and can only be determined by an artificial ſcale, aſſuming what effects ſhall be the meaſures of equal portions of heat.

To

Among the things, which have been thus ſubjected to menſuration, are *Ratios.* Analogy in ſome degree points out the ſcale, but not ſo clearly as to ſupercede the neceſſity for a definition; for though it readily appears, that ratios ſimilar according to the definition in art. 252 muſt be put equal in magnitude, it does not readily appear what alteration in the terms of a ratio ought to be conſidered as doubling it or tripling it. On the firſt conſideration it might ſeem, that of two magnitudes compared with a common ſtandard, if one contained the ſtandard twice as often as the other, its ratio to the ſtandard ſhould be called double the ratio of the other; if thrice, triple; &c. or, (which is the ſame thing) that the magnitude of a ratio ſhould be proportional to the magnitude of the fraction whoſe numera-

To take one inſtance more. Differently conſtituted angles excite different ideas, but are not of themſelves ſubject to meaſure; although the idea of an angle, and of the ſurface included between the lines which form the angle, are ſo connected, as to render it almoſt impoſſible to miſtake what is meant by equal angles, double angles, &c. The formal definition would be as follows: Two angles are ſaid to be equal when, if the two angular points be placed together, and one line of the one angle laid along one of the lines of the other angle, the remaining line of the former angle will lie along the remaining line of the latter angle. And one angle is ſaid to be double, triple or any other multiple of another angle, when by drawing one, or two, or any other number of lines from the angular point, between the two lines of the one angle, there may be formed two, or three, or any other number of angles, equal among themſelves, and to that other angle. (See MASERES' TRIGONOMETRY, p. 209.)

tor is the antecedent, and denominator the conſequent of the ratio.

Thus it might ſeem that the ratio 8 : 2 ſhould be called double the ratio 4 : 2, becauſe 8 contains 2 twice as often as 4 does. But theſe conſiderations are perfectly nugatory; for the ſole uſe and intention of a definition of double, triple, &c. ratios is to expedite mathematical reaſoning; and this intention will be found to be beſt conſulted by a definition very different to that which would reſult from the above conſiderations.

282. *Def.* Ratios, ſimilar according to the definition in art. 252, are equal in magnitude.

283. *Def.* If there be any number of continued proportionals, the magnitude of the ratio of the firſt term to the third is double of the magnitude of the ratio of the firſt term to the ſecond; and the magnitude of the ratio of the firſt term to the fourth is triple of the magnitude of the ratio of the firſt term to the ſecond; and ſo on. And converſely, the magnitude of the ratio of the firſt term to the ſecond is half the magnitude of the ratio of the firſt term to the third; and a third part of the magnitude of the ratio of the firſt term to the fourth; and ſo on.*

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* This definition might have been given as follows: "The ratio of a to $\frac{m}{n}$ths of $\frac{m}{n}$ths of a is double of the ratio of a to $\frac{m}{n}$ths of a;

Thus, if a, b, c, &c. be continued proportionals, or $a : b :: b : c :: c : d :: d : e ::$ &c. the ratio $(a:c) = 2\times(a:b)$; $(a:d) = 3\times(a:b)$; $a:e = 4\times(a:b)$; &c. and $a : b = \frac{1}{2}\times(a:c) = \frac{1}{3}\times(a:d) =$ &c.

If t be put to denote the ratio $a : b$, then $(a:c) = 2t$; $(a:d) = 3t$; &c.

284. The double, triple, &c. of a ratio (according to the above definition) is the ſame as the duplicate, triplicate, &c. of the ſame ratio, according to the definition in art. 279. Hence, by art. 280, the double of any ratio is equal to the ratio between the ſquares of the terms; and the triple of any ratio is equal to the ratio between the cubes of the terms; &c. And converſely, the half of any ratio is equal to the ratio between the ſquare roots of the terms; and the third part of any ratio is equal to the ratio between the cube roots of the terms; &c.

Thus twice the ratio $2 : 3$ is equal to the ratio $4 : 9$; and three times the ratio $2 : 3$ is equal to the ratio $8 : 27$; and 4 times the ratio $2 : 3$ is equal to the ratio $16 : 81$; and m times the ratio $2 : 3$ is equal to the ratio $2^m : 3^m$.

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a; and the ratio of a to $\frac{m}{n}$ths of $\frac{m}{n}$ths of $\frac{m}{n}$ths of a is triple of the ratio of a to $\frac{m}{n}$ths of a; &c."

For if $a : b :: b : c ::$ &c. and $b = \frac{m}{n}\times a$, $c = \frac{m}{n}\times b = \frac{m}{n}\times\frac{m}{n}\times a$; and $d = \frac{m}{n}c = \frac{m}{n}\times\frac{m}{n}\times\frac{m}{n}\times a$; &c.

It appears from art. 274, that the like multiples of two *equal* ratios will be equal, and that the like multiples of two *unequal* ratios will be *unequal;* therefore the definition is conſiſtent with itſelf.

285. From the above definition any part, multiple or parts of any propoſed ratio may be determined. For example, if three halves of the ratio $a : b$ be required, let r be a mean proportional between a and b, and s a third proportional to r and b; then $a : s$ is the ratio required. For if $a : r :: r : b :: b : s$, $(a : r) = \frac{1}{2}(a : b)$ by definition; and $(a : s) = 3 \times (a : r)$.

$\therefore (a : s) = 3 \times \frac{1}{2}(a : b) = \frac{3}{2} \times (a : b)$.

And in the ſame manner $\frac{m}{n}$ths of the ratio $a : b$ is found by inſerting $\overline{n-1}$ mean proportionals between a and b, and continuing theſe proportionals (if neceſſary) till there be m ratios; then the ratio of a to the conſequent of the mth ratio is the ratio required, or $\frac{m}{n}$ths of $a : b$.

286. If $a : r :: r : b :: b : s$, $a : r = (a^{\frac{1}{2}} : b^{\frac{1}{2}})$; and $(a : s) = (a^{3} : r^{3}) = (\overline{a^{\frac{1}{2}}}|^{3} : \overline{b^{\frac{1}{2}}}|^{3}) = (a^{\frac{3}{2}} : b^{\frac{3}{2}})$ i. e. three halves of any ratio is equal to the ratio of the $\frac{3}{2}$ power of the terms. *In general* $\frac{m}{n}$ths of any ratio $a : b$ is equal to the ratio $a^{\frac{m}{n}} : b^{\frac{m}{n}}$. For

one nth of the ratio $a : b$ (or the ratio of a to the firſt of $\overline{n-1}$ mean proportionals between a and b) is equal to $a^{\frac{1}{n}} : b^{\frac{1}{n}}$, and m times the ratio $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is equal to the ratio $a^{\frac{m}{n}} : b^{\frac{m}{n}}$; but ſince $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is an nth of $a : b$, m times $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is $\frac{m}{n}$ths of $a : b$.

$$\therefore\ \frac{m}{n}\times(a : b) = a^{\frac{m}{n}} : b^{\frac{m}{n}}.$$

For example, two thirds of the ratio $8 : 27$ is $8^{\frac{2}{3}} : 27^{\frac{2}{3}}$, or $4 : 9$.

287. Any root of a whole number or improper fraction is greater than unity. But ſo high a root may be taken, that the difference between it and unity ſhall be leſs than any propoſed quantity.

The firſt part of the article is evident, and the latter part may be proved as follows:

Let a denote any whole number; and if it be poſſible let $\frac{1}{r}$ be ſo ſmall a fraction, that there can be no root of a but what ſhall exceed $1+\frac{1}{r}$. Then ſince $a^{\frac{1}{n}}$ *muſt* neceſſarily exceed $1+\frac{1}{r}$ howſoever great n be taken, a *muſt* neceſſarily exceed $\overline{1+\frac{1}{r}}\Big|^{n}$ or $1 + nr +$ &c. but whatever r and a be, n may be taken ſo great that $1 + nr +$ &c. ſhall exceed a, which is contrary to the ſuppoſition; therefore the ſuppoſition is abſurd; and $a^{\frac{1}{n}}$ may be leſs than $1+\frac{1}{r}$.

Hence $\overline{\frac{a+b}{a}}\Big\rceil^{\frac{1}{n}}$ *may* be leſs than $1+\frac{1}{r}$; for $\overline{a+b}\rceil^{\frac{1}{n}}$ may; and $\overline{\frac{a+b}{a}}\Big\rceil^{\frac{1}{n}}$ is leſs than $\overline{a+b}\rceil^{\frac{1}{n}}$.

288. Any root of a proper fraction is leſs than unity; but ſo high a root may be taken, that the difference between it and unity ſhall be leſs than any propoſed quantity.

For let $\frac{a}{a+b}$ denote any proper fractions; then ſince $\overline{\frac{a+b}{a}}\Big\rceil^{\frac{1}{n}}$ may be leſs than $\frac{r}{r-1}$ howſoever great r be taken, $\overline{\frac{a}{a+b}}\Big\rceil^{\frac{1}{n}}$ may be greater than $\frac{r-1}{r}$ or $1-\frac{1}{r}$.

289. The leſs any ratio is the nearer do its terms approach to equality; and ſo ſmall a part of any propoſed ratio may be taken, that the difference of the terms ſhall be leſs than any propoſed part of either of them.

For let any ratio be denoted by $a:b$; then an *n*th part of the ratio, or the ratio of a to the firſt of $n - 1$ mean proportionals between a and b, is equal to $a^{\frac{1}{n}} : b^{\frac{1}{n}}$, but n may be taken ſo great, that the difference between $a^{\frac{1}{n}}$ and 1, or $b^{\frac{1}{n}}$ and 1 ſhall be leſs than any propoſed quantity; conſequently n may be taken ſo great, that the difference between $a^{\frac{1}{n}}$ and $b^{\frac{1}{n}}$ ſhall be leſs than any propoſed part of $a^{\frac{1}{n}}$ or of $b^{\frac{1}{n}}$.

290. Cor. The magnitude of a ratio of equality, or ratio, whoſe antecedent and conſequent are equal, is nothing. For o is the limit to which any magnitude approaches by being continually decreaſed, and equality, it appears by the above, is the limit to which the terms of a ſmall part of any ratio approach, by taking that part to be leſs and leſs continually.*

291. Ratios are of two kinds; Ratios *majoris inæqualitatis*, or where the antecedent is greater than the conſequent; and Ratios *minoris inæqualitatis*, or where the antecedent is leſs than the conſequent.

292. Any part, multiple or parts of a ratio *majoris inæqualitatis* is alſo a ratio *majoris inæqualitatis;* and any part, multiple or parts of a ratio *minoris inæqualitatis* is alſo a ratio *minoris inæqualitatis.*

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* Aliter. Let $a : a = d$; then $a^n : a^n = nd$; but $a^n : a^n = a : a$ $\therefore nd = d$ $\therefore d = o$.

For let $a : b$ denote any propoſed ratio; and m $n$ths denote any multiple, part or parts.

Then if a be greater than b, $a^{\frac{m}{n}}$ is greater than $b^{\frac{m}{n}}$, and if a be leſs than b, $a^{\frac{m}{n}}$ is leſs than $b^{\frac{m}{n}}$; but $a^{\frac{m}{n}} : b^{\frac{m}{n}}$ is equal to m $n$ths of the ratio $a : b$.

293. Hence it appears, that if any poſitive quantity, t, be aſſumed as the meaſure of the magnitude of ſome certain ratio *majoris inæqualitatis*, $a+c : a$, any *other* propoſed poſitive quantity will be the meaſure of ſome other ratio *majoris inæqualitatis*, as being ſome part, multiple or parts of t.

For example, let s denote any other poſitive quantity; then s being $\frac{s}{t}$ parts of t is the meaſure of a ratio which is $\frac{s}{t}$ parts of $a + c : a$, or of the ratio $\overline{a+c}|^{\frac{s}{t}} : a^{\frac{s}{t}}$. That is, ratios *majoris inæqualitatis* employ for their meaſures all poſſible degrees of magnitude from the leaſt to the greateſt. And in the ſame manner it appears, that if a poſitive quantity be aſſumed as the meaſure of any ratio *minoris inæqualitatis*, any other propoſed poſitive quantity will be the meaſure of ſome other ratio *minoris inæqualitatis*. Or ratios *minoris inæqualitatis* do alſo employ for their meaſures all poſſible degrees of magnitude from the leaſt to the greateſt. So that if poſitive quantities were put for the meaſures of both kinds of ratios, this would happen—viz. that one meaſure would belong to two ratios. Hence it is found convenient to aſſume poſitive quantities as the meaſures of one kind of ratios, and negative quantities as meaſures of the * other kind; according to the following definition:

294. *Def.* The magnitudes of ratios *majoris inæqualitatis* are meaſured by poſitive quantities, and the magnitudes of ratios *minoris inæqualitatis* are meaſured by negative quantities; and that quantity, which, when taken poſitively, is aſſumed the meaſure of the ratio $a + c : a$, when taken negatively is the meaſure of the ratio $a : a + c$. †

295. Cor. To add any ratio is the ſame as to ſubtract the inverſe ratio; and to ſubtract any ratio is the ſame as to add the inverſe ratio.

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* It is entirely arbitrary which kind of ratio be meaſured by poſitive quantities, and which by negative.

† It would be conformable with this and the previous definitions to call ratio, "The (relative) *preponderance* of one quantity over another," for when the preponderance of the antecedent *vaniſhes*, the magnitude of the ratio *vaniſhes*, (ſee art. 290), and when, by a ſtill further diminution of the antecedent, the *preponderance* is converted into *deficiency*, or inclines to the *other* term, (by this definition) the magnitude of the ratio becomes *negative*.

296. If $a : b$ be poſitive, $a^{-\frac{m}{n}} : b^{-\frac{m}{n}}$ is negative. For $a^{-\frac{m}{n}} : b^{-\frac{m}{n}} = b^{\frac{m}{n}} : a^{\frac{m}{n}}$, which is a ratio *minoris inæqualitatis*, if a be greater than b, (art. 292.) And in a ſimilar manner it appears, that if $a : b$ be negative, $a^{-\frac{m}{n}} : b^{-\frac{m}{n}}$ is poſitive.

297. If two ratios be compounded according to the definition in art. 277, the reſulting ratio is the *Algebraic ſum* of the two ratios; i. e. when the two ratios are both of the ſame kind the reſulting ratio is a greater ratio of the ſame kind, whoſe magnitude is the ſum of the magnitudes of the two ratios; and when the two ratios are of oppoſite kinds, the reſulting ratio is of the ſame kind as that, whoſe abſtract magnitude is the greater (or whoſe inverſe ratio is greater than the other ratio) and its magnitude is the difference of the abſtract magnitudes of the two ratios. For,

1ſt. Let the two ratios be of the ſame kind, both poſitive or both negative, and equal to the ratios $a : b$ and $c : d$; then their compound ratio is equal to $ac : bd$ (art. 275). Now $c : d$ is ſome multiple, part or parts of $a : b$; let it be $\frac{m}{n}$ths; then $c : d :: a^{\frac{m}{n}} : b^{\frac{m}{n}}$ and $ac : bd = a \times a^{\frac{m}{n}} : b \times b^{\frac{m}{n}} = a^{1+\frac{m}{n}} : b^{1+\frac{m}{n}} = \overline{1 + \frac{m}{n}} \times (a : b) = (a : b) + \frac{m}{n} \times (a : b) = (a : b) + (c : d)$

2dly. Let the two ratios be of different kinds, and equal to $a : b$ and $e : f$; and let $a : b$ be the greater of the two, i. e. greater than $f : e$; then $f : e$ is ſome part or parts of $a : b$; let it be $\frac{m}{n}$ths; then $f : e :: a^{\frac{m}{n}} : b^{\frac{m}{n}}$ and $e : f :: a^{-\frac{m}{n}} : b^{-\frac{m}{n}}$; $\therefore\ ae ; bf = a^{1-\frac{m}{n}}\ b^{1-\frac{m}{n}}$, which is a ratio of the ſame kind as $a : b$, becauſe, by ſuppoſition, $\frac{m}{n}$ is leſs than 1, and conſequently $1 - \frac{m}{n}$ a poſitive index (ſee art. 292); but $a^{1-\frac{m}{n}} : b^{1-\frac{m}{n}} = \overline{1 - \frac{m}{n}} \times (a:b)$ $= (a:b) - \frac{m}{n} \times (a:b) = (a:b) - (f:e) = (a:b) + (e:f)$.

298. Cor. Hence to add ratios, multiply the antecedents together for a new antecedent, and the conſequents together for a new conſequent.

Ex. $(2 : 3) + (2 : 3) = 4 : 9$; $(1 : 2) + (2 : 3) = (2 : 6)$; $(3 : 5) + (5 : 3) = (15 : 15) = 0$.

299. Cor. 2d. Converſely, to ſubtract one ratio from another, take the quotient of the antecedents for a new antecedent, and the quotient of the conſequents for a new conſequent,—making the terms of the ratio to be ſubtracted the diviſors. Or add the inverſe of the ratio to be ſubtracted.

Ex. $(4 : 9) - (2 : 3) = (\frac{4}{2} : \frac{9}{3}$ or$)\ 4 \times 3 : 9 \times 2)$; $(5 : 4) - (1 : 2) = (\frac{5}{1} : \frac{4}{2}$ or$)\ 60 : 4$; $(1 : 2) - (2 : 1) = 1 : 4$; $(3 : 1) - (3 : 1) = (1 : 1) = 0$.

300. A ratio is multiplied by any whole number or fraction, $\pm \frac{m}{n}$, by raiſing its terms to the power of $\pm \frac{m}{n}$.

For let $(a : b) = \pm t$; then $(a^{\frac{m}{n}} : b^{\frac{m}{n}} = \frac{m}{n} \times \pm t$, (art. 286); and $(a^{-\frac{m}{n}} : b^{-\frac{m}{n}}) = (b^{\frac{m}{n}} : a^{\frac{m}{n}}) = \frac{m}{n} \times \mp t = - \frac{m}{n} \times \pm t$.

301. Converſely, a ratio is divided by any whole number or fraction, $\pm \frac{m}{n}$, by taking the $\pm \frac{m}{n}$ root of its terms, i. e. by raiſing them to the power $\pm \frac{n}{m}$.

For if $(a : b) = \pm t$, $(a^{\pm \frac{n}{m}} : b^{\pm \frac{n}{m}}) = \pm \frac{n}{m} \times \pm t$ (by laſt art.) $= \pm t \div \pm \frac{m}{n}$.

302. If two poſitive ratios have a common conſequent, that is the greater of the two, which has the greater antecedent; and if two negative ratios have a common antecedent, that is the greater of the two, which has the greater conſequent. (See art. 289).

Thus, 5 : 3 is greater than 4 : 3, and 3 : 5 is greater than 3 to 4.

303. If negative ratios be conſidered as *leſs* than poſitive ratios, and of two negative ratios, that be conſidered as the leſs, which is moſt removed from poſitive ratio, or whoſe inverſe ratio is leſs than the inverſe of the other ratio, then, *univerſally*, if two ratios have a common conſequent, that is the greater of the two, which has the greater antecedent.

Thus, for example, according to this conſideration, the ratio 4 : 6 is greater than 2 : 6, and the ratio 11 : 10 greater than 1 : 10.

304. A ratio is leſſened by having the ſame quantity added to both its terms; and increaſed by having the ſame quantity ſubtracted from both its terms. For let the terms be a, and $a + c$; then if to a there be added x, and to $a + c$ there be added a quantity whoſe ratio to x is the ſame as the ratio of $a + c$ to a, the ratio will not be altered. Conſequently if only x be added to $a + c$, the ratio will be leſſened, (art. 289).

Again, if from a and $a + c$ there be ſubtracted two quantities in the ratio of a to $a + c$, the ratio will not be altered; therefore if from $a + c$, the greater term, there be ſubtracted only as much as from a, the leſs, the ratio will be increaſed.

305. If negative be conſidered as *leſs* than poſitive ratios, a negative ratio may be ſaid to be *increaſed* by the addition of equal quantities to both terms, and *decreaſed* by the ſubtraction.

SCHOLIUM.

306. If, inſtead of the definition in article 283, the following definition had been aſſumed—"*A compound ratio is the* ſum *of the ratios compounded,*" it would have followed, without any further definition, that one kind of ratio muſt be poſitive and the other kind negative. For ſuch a definition would evidently have included that in article 283, and conſequently would have determined the magnitude of a ratio of equality to have been o. Since then it would have defined the ſum of $a:b$ and $b:a$ to be $ab:ba$, and would have determined $ab::ba$ to be o, it would have determined $(a:b)$ to be equal to $-(b:a)$.

But ſuch a definition, although more comprehenſive than that in art. 283, would not have been ſo advantageous in point of perſpicuity, as it would have confounded under one name both addition and ſubtraction. Nor would it have admitted the ſubject to have been treated with more brevity; as it would ſtill have been neceſſary to have reſolved it into the definition in art. 283, in order to have diſcovered what were meant by the double of a ratio, the triple of a ratio, &c. and in order to have found whether it were conſiſtent with itſelf.

CHAP.

CHAPTER X.

Of PERMUTATIONS and COMBINATIONS.

307. THE different poſitions with reſpect to order in which things may be placed, are called their Permutations.

Thus the Permutations of *a*, *b*, *c*, taken two and two together, are *ab*, *ba*, *ac*, *ca*, *bc*, *cb*.

308. The different collections that can be formed out of things, without regarding the order in which they are placed, are called their Combinations.

Thus, the combinations of *a*, *b*, *c*, taken two and two together, are *ab*, *ac*, *bc*.

309. The number of things to be joined together in one parcel or collection, is called the Exponent of the Combination.

Thus, if things be combined three and three together, the exponent of the combination is 3.

310. The number of Permutations, which *n* things, *a*, *b*, *c*, &c. all diſtinguiſhable from each

other, admit of, taken two and two together, is $n . \overline{n-1}$; taken three and three together, is $n . \overline{n-1} . \overline{n-2}$; taken four and four together is $n . \overline{n-1} . \overline{n-2} . \overline{n-3}$; and in general the number of Permutations which n things admit of, taken r and r together, is $n . \overline{n-1} . \overline{n-2} \ldots\ldots \overline{n-r-2} . \overline{n-r-1}$.

Demonſtration. Any one of the n quantities may ſtand firſt; ſo that there are n different ways of filling the firſt place.

Now which ever of the n things ſtand firſt, there will be the remaining $\overline{n-1}$ things to make choice of, in order to fill the ſecond place; i. e. there will be $\overline{n-1}$ ways of filling the ſecond place, when a ſtands firſt; $\overline{n-1}$ ways when b ſtands firſt; and ſo on, through all the n things. Conſequently there are $n . \overline{n-1}$ ways, in which the two firſt places may be filled; or $n . \overline{n-1}$ permutations of n things taken two and two together.

Now whichever of theſe $n . \overline{n-1}$ ways of filling the two firſt places be adopted, there will be $\overline{n-2}$ things remaining to make choice of, in order to fill the third place; i. e. there will be $\overline{n-2}$ ways of filling the third place when ab ſtands firſt; $\overline{n-2}$ ways when ba ſtands firſt; and ſo on, through all the $n . \overline{n-1}$ quantities *ab*, *ba*, *ac*, *ca*, &c. Conſequently there are $n . \overline{n-1} . \overline{n-2}$ different ways in which the three firſt places may be filled; or $n . \overline{n-1} . \overline{n-2}$ permutations of n things taken three and three together. And, in the ſame manner, becauſe whichever of theſe $n . \overline{n-1} . \overline{n-2}$ ways be adopted in filling the three firſt places, there will remain $\overline{n-3}$ things to make choice of in order to fill the fourth place, it follows that there are $n . \overline{n-1} . \overline{n-2} . \overline{n-3}$ different ways of filling the four firſt places; or $n . \overline{n-1} . \overline{n-2} . \overline{n-3}$ permutations of n things taken four and four together, &c.

EXAMPLES.

The number of permutations which ſix different things, a, b, c, d, e, f, admit taken 4 and 4 together, is $6 \times 5 \times 4 \times 3$ or 360.

The number of permutations which three things a, b, c, admit, taken all together, is $3 \times 2 \times 1$ or 6.

The number which four things a, b, c, d admit, taken all together, is $4 \times 3 \times 2 \times 1$ or 24.

The number of changes which can be rung on ſix bells, is $6 \times 5 \times 4 \times 3 \times 2 \times 1$ or 720.

311. The number of combinations that can be formed out of n things, taken *two and two* together, is $n . \frac{n-1}{2}$; taken *three* and *three* together,

is $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$; taken *four* and *four* together, is $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$; taken r and r together, is $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \frac{n-\overline{r-1}}{r}$.

For each parcel or combination, conſiſting of two things, admits of two permutations; therefore there are twice as many permutations as combinations of n things taken *two* and *two* together; but the number of permutations is $n \cdot \overline{n-1}$; therefore the number of combinations is $n \cdot \frac{n-1}{2}$.

Again, each parcel or combination, conſiſting of three things, admits of $3 \cdot 2 \cdot 1$ permutations; (art. 310) therefore there are $3 \cdot 2 \cdot 1$ times as many permutations as combinations of n things taken *three* and *three* together; but the number of permutations is $n \cdot \overline{n-1} \cdot \overline{n-2}$; therefore the number of combinations is $\frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{1 \cdot 2 \cdot 3}$.

In general, Each parcel or combination conſiſting of r things, admits of $r \cdot \overline{r-1} \cdot \overline{r-2} \cdot\cdot\cdot\cdot\cdot 2 \cdot 1$ permutations (art. 310); therefore there are $r \cdot \overline{r-1} \cdot \overline{r-2} \cdot\cdot\cdot\cdot 2 \cdot 1$ times as many permutations as combinations of n things taken r and r together; but the number of permutations is $n \cdot \overline{n-1} \cdot \overline{n-1} \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \overline{n-\overline{r-1}}$; therefore the number of combinations is $\frac{n \cdot \overline{n-1} \cdot \overline{n-2} \cdots\cdots \overline{n-\overline{r-1}}}{1 \cdot 2 \cdot 3 \cdot\cdot\cdot\cdot\cdot\cdot r}$

312. The number of combinations of n things is the ſame when taken * *one* at a time, as when taken *all but one* at a time; when taken *two* at a time, as when taken *all but two* at a time; and ſo on. For there evidently muſt be the ſame choice in electing one of the n things, as in omitting one of them; in electing pairs of them as in omitting pairs; &c. It is evident alſo from the conſtruction of the above expreſſions; for

$$\frac{n.\overline{n-1}\ldots 3.2.1}{1.2.3\ldots\overline{n-2}.\overline{n-1}.n}=1\text{; and }\frac{n.\overline{n-1}\ldots 3.2}{1.2.3\ldots\overline{n-2}.\overline{n-1}}=n$$

$$\text{and }\frac{n.\overline{n-1}\ldots 3}{1.2.3..\overline{n-2}}=\frac{n.\overline{n-1}}{2}\text{; \&c.}$$

313. The total number of combinations that can be formed of n different things according to all the exponents they will admit of, (i. e. the ſum of all the combinations of the ſingle things, of all the combinations of two, of all the combinations of three, &c.) is $2^n - 1$.

For a ſingle thing, a, forms one combination; the introduction of a ſecond, b, adds two more, —viz. b by itſelf and ab; the introduction of a third, c, adds four more,—viz. c by itſelf and c

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* When things are taken *ſingly*, they cannot, in a ſtrict ſenſe, be ſaid to be *combined*; but it is convenient to include this caſe under the general name, for the ſake of avoiding a multiplicity of words.

For the like reaſon it is convenient to conſider *the total omiſſion of all* the things as one ſpecies of combination.

combined with each of the former three; the introduction of a fourth, *d*, adds eight more,—viz. *d* by itſelf and *d* combined with each of the former ſeven,* and in general the introduction of a new letter adds as many new combinations as there were combinations without it, and one beſides. Hence, the demonſtration may be reduced to this —To prove that in the ſeries 1, 2, 4, 8, 16, &c. whoſe firſt term is 1, and each ſucceeding term of which is the *ſum of all the preceding* + 1, the 3d ſum of *n* terms will be 2^{n-1}.

Now that this is true at firſt, will appear evident, from computing the ſums of the terms; for $1=2^1-1$; $1+2=3=2^2-1$; $1+2+4=7=2^3-1$; $1+2+4+8=15=2^4-1$; &c. therefore, if it can be ſhewn to be a neceſſary conſequence of its obtaining in the ſum of *r* terms, that it muſt alſo obtain in the ſum of $r+1$ terms, or in other words

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[*] If the additional combinations formed by the introduction of each new letter, be ſet down in a ſeparate row, they will ſtand thus:

a

b, *ab*

c, *ac*, *bc*, *abc*

d, *ad*, *bd*, *cd*, *abd*, *acd*, *bcd*, *abcd*

e, *ae*, *be*, *ce*, *de*, *abe*, *ace*, *bce*, *ade*, *bde*, *cde*, *abce*, *abde*, *acde*, *bcde*, *abcde*, &c.

Where each row will have as many combinations *in it as there are in all the preceding*, *and one beſides*.

if it can be ſhewn that, provided the ſum of r terms be $2^r - 1$, the ſum of $\overline{r+1}$ terms is $2^{r+1} - 1$, it will be eſtabliſhed, generally, that the ſum of n terms is $2^n - 1$. Now this may be done as follows: Let the ſum of r terms be $2^r - 1$, then the *next* term by the conſtruction of the ſeries is $2^r - 1 + 1$, or 2^r; therefore the ſum of $r+1$ terms is $2^r - 1 + 2^r = 2 \times 2^r - 1 = 2^{r+1} - 1$. It is true then in general, that the ſum of n terms is $2^n - 1$. Therefore the total number of combinations of n things is $2^n - 1$.

Ex. The number of different ſums that may be formed with a guinea, a half guinea, a crown, a half crown, a ſhilling, and a ſixpence, is $2^6 - 1$, or 63.

314. Cor. Hence, ſince n is the number of ſingle things, $n \,.\, \frac{n-1}{2}$ the number of combinations by twos, $n \,.\, \frac{n-1}{2} \,.\, \frac{n-2}{3}$ the number of combinations by threes, &c. (art. 311), it follows that,

$$n + n.\frac{n-1}{2} + n.\frac{n-1}{2}.\frac{n-2}{3} + \&c. \text{ to } n \text{ terms} = 2^n - 1$$

315. Cor. 2d. $1 + n + n.\frac{n-1}{2} + \&c.$ to $\overline{n+1}$ terms $= 2^n$.

Cor. 3d.

316. Cor. 3d. The laſt term of the ſeries $n + n.\frac{n-1}{2}$ + &c. to n terms, is equal to 1 (art. 312); hence $n + n.\frac{n-1}{2} + \&c.$ to $\overline{n-1}$ terms $= 2^n - 2$.

and (dividing by n) $1 + \frac{n-1}{2} + \frac{n-1}{2}..\frac{\overline{n-2}}{3} + \&c.$ to $n-1$ terms $= \frac{2^n-2}{n}$.

317. If there be n ſets of things, all different, and the firſt ſet conſiſt of p things, and the ſecond ſet conſiſt of q things, &c. then the number of parcels of n things in each, that may be formed by ſelecting one thing out of each of the n ſets, is equal to the product of the numbers p, q, &c.

For let the 1ſt ſet be $a, b, c,$ &c. (p things)
and the 2d . . . be $\overset{\prime}{a}, \overset{\prime}{b}, \overset{\prime}{c},$ &c. (q things)
and the 3d . . . be $\overset{\prime\prime}{a}, \overset{\prime\prime}{b}, \overset{\prime\prime}{c},$ &c. (r things)
&c. &c. to n ſets.

Then in the 1ſt ſet there will be the choice of p things. And to any one of theſe p things there may be united any one of the q things in the 2d ſet, i. e. to a there may be united $\overset{\prime}{a}$, or $\overset{\prime}{b}$, or $\overset{\prime}{c}$, &c. to b there may be united $\overset{\prime}{a}$, or $\overset{\prime}{b}$, or $\overset{\prime}{c}$, &c. and ſo on through all the p things; conſequently the number of parcels of two things each, that may be ſelected from the two firſt ſets, is $p \times q$. And to any one of theſe pq parcels there may be united any one of the r things in the 3d ſet; conſequently the number of parcels of three things each, that may be ſelected from the three firſt ſets, is $p \times q \times r$. &c. &c.

318. Cor. If each ſet conſiſt of the *ſame* number of things, or $p = q = r =$ &c. The number of parcels is $p \times p \times p \times$ &c. to n factors, $= p^n$.

319. Cor. 2d. Let there be two things in each ſet; and let the 1ſt ſet be a, b; the 2d, $\acute{a}$, $\acute{b}$; the 3d, a'', b''; and ſo on to n ſets. Then,

1ſt. It is evident that there will be one parcel compoſed entirely of the a's.

2dly. There will be n parcels compoſed of $\overline{n-1}$ a's and one b. For any one of the a's may be omitted; and there are n to make choice of.

3dly. There will be $n \,.\, \frac{n-1}{2}$ parcels compoſed of $\overline{n-2}$ a's, and *two* b's. For any pair of the a's may be omitted, and the number of pairs in n things is $n \,.\, \frac{n-1}{2}$.

4thly. In the ſame manner it appears that there will be $n \,.\, \frac{n-1}{2} \,.\, \frac{n-2}{3}$ parcels compoſed of $\overline{n-3}$ a's, and *three* b's, and ſo on.

Hence, if the parcel conſiſting of the n a's be denoted by a^n; and any one of the parcels conſiſting of $\overline{n-1}$ a's and *one* b, be *indifferently* denoted by $a^{n-1}b$; and ſo on; the parcels will be

$$a^n + n\, a^{n-1}b + n . \frac{n-1}{2} \times a^{n-2}b^2 + \text{\&c. to } n+1 \text{ terms.}$$

EXAMPLES.

EXAMPLES.

Out of four companies three of which conſiſt of five men each, and the other of ſeven men, there may be ſelected a party of four men, one from each company ($5 \times 5 \times 5 \times 7$ or) 875 different ways.

Ex. 2d. A frame, containing n ſhutters, each of which admits of p different poſitions, all diſtinguiſhable from one another, may exhibit p^n different ſignals.*

For the frame will exhibit all the ſhutters at once in one or other of the different poſitions; therefore the number of ſhutters correſponds to the number of ſets in the above theorem, and the number of different poſitions which each ſhutter admits, correſponds to the number of things in each ſet.

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* If $n = 6$ and $p = 2$, the caſe becomes that of the *Telegraph*, according to its preſent conſtruction at the Admiralty-Office; conſequently the number of ſignals which that machine can exhibit is 2^6 or 64.

If one of the poſitions be conſidered as permanent during the intermiſſion of ſignals, the number of ſignals is but 63.

CHAP.

CHAPTER XI.

OF THE BINOMIAL THEOREM.

320. SIR ISAAC NEWTON's Binomial Theorem is a propoſition affirming that, if *m* be any number whatſoever, either integral or fractional, poſitive or negative, and *b* be either poſitive or negative, $\overline{a+b}\rceil^{m}$ is truly expreſſed by the ſeries,

$$a^m + m\times a^{m-1}b + m.\frac{m-1}{2}\times a^{m-2}b^2 + m.\frac{m-1}{2}.\frac{m-2}{3}\times a^{m-3}b^3 + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}.a^{m-4}b^4$$

+ &c. Where the indices of *a* beginning from *m*, decreaſe by unity, and the indices of *b* beginning from 0, increaſe by unity, in each ſucceeding term; and where the coefficient of the *n*th term (*n* denoting any number) is the coefficient of the $\overline{n-1}\rceil^{\text{th}}$ term multiplied into $\frac{m-\overline{n-2}}{n-1}$.

If *m* be a poſitive whole number, the ſeries will conſiſt of but *n* + 1 terms, each ſucceeding coefficient after that term involving the factor $\frac{m-m}{m+1}$, which is 0; but in every other caſe, the new factor introduced into the coefficient will always have either a poſitive or a negative value, and the number of terms will be unlimited.

321. But in order that no miſtake may be made concerning what it is, that the Binomial theorem aſſerts in thoſe caſes, where the ſeries is unlimited, it may not be improper to conſider what is meant by a ſeries expreſſive of roots, roots of powers, or their reciprocals.

322. In the firſt place then, the *n*th root of any binomial, $a+b$, is that ſeries, of which if any propoſed number of terms be taken, and their ſum be raiſed to the *n*th power, two of the terms in the reſult will be a and b, and of the remaining powers of a and b as many of the coefficients as are complete, i. e. as would not be affected by taking in more terms of the ſeries, will conſiſt of poſitive and negative terms which mutually deſtroy each other. And ſuch a ſeries, may be found in *any propoſed* caſe of n, by the rule in art. 200, or if $n = 2$ by the rule in art. 196, and if $n = 3$ by the rule in art. 198.

But ſince $\overline{a+b}\rceil^{\frac{1}{n}} = a^{\frac{1}{n}} \times \overline{1+\frac{b}{a}}\rceil^{\frac{1}{n}}$ and ſince the ſeries that would be found by the rule in art. 200, for $\overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}}$, if multiplied into $a^{\frac{1}{n}}$ would become the *ſame* as the ſeries found for $\overline{a+b}\Big|^{\frac{1}{n}}$ by that ſame rule, it will render this inquiry more ſimple, to confine it *at firſt* to ſuch binomials, as have *unity* for one of their terms.

323. Let then $1+x$ denote any ſuch binomial, and let its nth root be extracted by the rule in art. 200; and it is evident that the firſt term of the reſulting ſeries will be unity; and that the ſucceeding terms will be the ſucceſſive integral powers of x, with certain coefficients; i. e. if a, b, c, &c. be aſſumed to denote the coefficients of the 2d, 3d, 4th, &c. terms reſpectively, the ſeries will be $1+ax+bx^2+cx^3+dx^4+$ &c. and theſe coefficients will depend only on n; or there will be but one ſet in any one propoſed caſe of n. Now this ſeries, by its conſtruction, is to be of ſuch a nature, that if three of the terms be taken, and their ſum be raiſed to the nth power, the reſult will be of the form $1+x * +px^3+qx^4 \ldots\ldots +{}$† $b^n x^{2n}$, where the ſquare of x is wanting,—becauſe the coefficient of x^2 will be *complete*, or would not be

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† In the product $\overline{1+ax+bx^2}\times\overline{1+ax+bx^2}\times$ &c. to n factors, the term involving the higheſt power of x will evidently be $bx^2\times bx^2\times bx^2\times$ &c. to n factors, or $b^n x^{2n}$; and in the ſame manner in the product $\overline{1+ax+bx^2+cx^3}\Big|^n$ the term involving the higheſt power of x will be $c^n x^{3n}$; and ſo on.

affected by taking in one or more of the ſucceeding terms, cx^3, dx^4, &c.

For the ſame reaſon, if four terms be taken, and their ſum be raiſed to the nth power, the product, or $\overline{1 + ax + bx^2 + cx^3}\rceil^n$ will be of the form $1 + x * * + px^4 + qx^5 \ldots\ldots + c^n x^{3n}$, where both x^2 and x^3 are wanting. And in general if $\overline{r + 1}$ terms be taken, and their ſum be raiſed to the nth power, the product, or, (ſuppoſing D the coefficient of the $\overline{r+1}\rceil^{th}$ term)

$$\overline{1 + ax + bx^2 + cx^3 + dx^4 \ldots\ldots + Dx^r}\rceil^n$$

will be of the form

$$1 + x * * * * + px^{r+1} + qx^{r+2} \ldots\ldots + D^n x^{rn}.$$

324. Hence, as in diviſion (in thoſe caſes, where, how far ſoever the operation be carried on, a remainder muſt always be left) the ſeries obtained does or does not approximate to the exact value of the quotient, according as the remainders decreaſe or not; ſo in extracting roots, the ſeries for $\overline{1+x}\rceil^{\frac{1}{n}}$ will or will not approximate to the exact nth root of $1+x$, according as the ſeveral differences between $1+x$ and the nth power of the firſt three, four, &c. terms of the ſeries decreaſe or not. And if x be ſuch, that no quantity d can be aſſigned, ſo ſmall, but that ſo many terms of the ſeries may be taken, that the difference between the nth power of their ſum and $1+x$ ſhall be leſs than that quantity d, the accuracy of the approximation is unlimited.

Having

325. Having eſtabliſhed in the preceding articles, the nature of the ſeries for $\overline{1+x}\rceil^{\frac{1}{n}}$ it remains in the ſecond place to conſider the nature and form of the ſeries for $\overline{1+x}\rceil^{\frac{r}{n}}$ where r and n denote any two poſitive whole numbers whatſoever.

Now this ſeries is that which may be obtained either by raiſing $1+x$ to the rth power, and extracting the nth root of the reſult, (by the rule in art. 200), or by extracting the nth root of $1+x$, and raiſing the reſult to the rth power. Let then $\overline{1+x}^{r} = 1+px+qx^2 \ldots\ldots +x^m$; and let $\overline{1+x}^{\frac{1}{n}} = 1+ax+bx^2+cx^3+\&c.$ Then $\overline{1+x}^{\frac{r}{n}}$ is either $\overline{1+px+qx^2 \ldots\ldots +x^m}\rceil^{\frac{1}{n}}$ i. e. $1+a\times\overline{px+qx^2 \ldots +x^m} + b\times\overline{px+qx \ldots\ldots +x^m}\rceil^2 + c\times\overline{px+qx^2 \ldots\ldots +x^m}\rceil^3 + \&c.$; or $\overline{1+ax+bx^2+cx^3+\&c.}\rceil^r$ i.e. $1+p\times\overline{ax+bx^2+cx^3+\&c.} + q\times\overline{ax+bx^2+cx^3+\&c.}\rceil^2 \ldots\ldots + \overline{ax+bx^2+cx^3+\&c.}\rceil^m$

And it is evident that in either caſe, if the terms were worked out at length, the ſeries would have unity for its firſt term, and for the ſucceeding terms the ſucceſſive powers of x with certain coefficients. Moreover, ſince the nth root and rth power of any expreſſion have each but one ſeries, the nth root of $\overline{1+x}|^{r}$ and the rth power of $\overline{1+x}|^{\frac{1}{n}}$ will each have but one ſeries.

326. It appears from the preceding articles, that if m be put to denote any number whatſoever, integral or fractional, the ſeries for $\overline{1+x}|^{m}$ is of the form $1 + ax + bx^2 + cx^3 + \&c.$ . . and it remains in the third place to ſhew, that the ſame form obtains in the ſeries for $\overline{1+x}|^{-m}$; Now $\overline{1+x}|^{-m}$ or $\dfrac{1}{\overline{1+x}|^{m}}$ is that ſeries which is to be had by dividing unity by $\overline{1+x}|^{m}$; Let then $\overline{1+x}|^{m} = 1 + ax + bx^2 + cx^3 + \&c.$; and the operation will be as follows:

$$\begin{array}{ll} \text{Diviſor.} & \qquad\text{Dividend.} \qquad\qquad \text{Quotient.} \\ 1 + ax + bx^2 + cx^3 + \&c.) & 1*** \; (1 - ax + \overline{a^2 - b}.x^2 + \&c. \\ & 1 + ax + bx^2 + cx^3 + \&c. \\ \hline & \qquad - ax - bx^2 \quad - cx^3 \quad - \&c. \\ & \qquad - ax - a^2x^2 - abx^3 - \&c. \\ \hline & \qquad \overline{a^2 - b}\,.\,x^2 + \overline{ab - c}.x^3 + \&c. \end{array}$$

where the firſt term of the quotient is unity, and the ſucceeding terms, the ſucceſſive powers of x, with certain coefficients,

327. If, in the above articles, the ſecond term of the binomial had been taken negative, the ſame concluſions would have reſulted; i. e. the ſame

ſeries, which in any propoſed caſe expreſſes $\overline{1+x}\rceil^{m}$, will, by ſubſtituting $-x$ and its powers, for $+x$ and its powers, become the proper expreſſion for $\overline{1-x}\rceil^{m}$.

328. Let *m* denote any number whatſoever, integral or fractional, poſitive or negative, then the coefficient of the ſecond term in the ſeries for $\overline{1+x}\rceil^{m}$ is the index *m*.

For let $\overline{1+x}\rceil^{r}$ (*r* being a whole number) be $1+rx+$ &c.; then $\overline{1+x}\rceil^{r+1}$ is $\overline{1+rx+\&c.}\times\overline{1+x}$, or $1+\overline{r+1}\,.\,x+$ &c. i. e. (in the caſe of whole poſitive powers) if the coefficient of the ſecond term *be* the index in one inſtance, it will in the next inſtance, and conſequently will univerſally. But it is ſo in the ſquare—therefore it is ſo univerſally.

Hence, $\overline{1+x}^{\frac{1}{n}}$ will be $1+\frac{1}{n}\times x+$ &c. as appears from the rule in art. 200, where this law is * aſſumed to obtain in whole poſitive powers.

Hence, $\overline{1+n}^{\frac{r}{n}}$ will be $1+\frac{r}{n}\times x+$ &c.

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* This aſſumption is not eſſential to that rule; for the principle of the rule is only this—that *if* p (whatever p be) *be* the coefficient of the ſecond term of the nth power of a binomial, the conſtant diviſor muſt be p times the $\overline{n-1}$ power of the firſt term of the root. (See art. 199 and 200.)

For $\overline{1+x}\rceil^{\frac{r}{n}}$ is either $\overline{1+rx-\&c.}\rceil^{\frac{1}{n}}$ or $\overline{1+\frac{1}{n}x+\&c.}\rceil^{r}$

but $\overline{1+rx+\&c.}\rceil^{\frac{1}{n}}$ (by the preceding caſe) is $1+\frac{1}{n}\times\overline{rx+\&c.}+\&c.=1+\frac{r}{n}.x+\&c.$ And $\overline{1+\frac{1}{n}.x+\&c.}\rceil^{r}$ is $1+r\times\overline{1+\frac{1}{n}.x+\&c.}+\&c.$ $=1+\frac{r}{n}.x+\&c.$

Hence, ſince the coefficient of the ſecond term is univerſally the index in all caſes of poſitive powers, it is alſo in all caſes of negative powers, as appears from the work in art. 326, where a being aſſumed the coefficient of x in the poſitive power, $-a$ is found to be its coefficient in the reciprocal of that power.

329. If $\overline{1+x}\rceil^{m}=1+mx+bx^2+cx^3+\&c.$ $\overline{a+x}\rceil^{m}=a^m+ma^{m-1}x+ba^{m-2}x^2+\&c.$

For $\overline{a+x}\rceil^{m}=a^m\times\overline{1+\frac{x}{a}}\rceil^{m}=a^m\times$ $\overline{1+m.\frac{x}{a}+b.\frac{x^2}{a^2}+c.\frac{x^3}{a^3}+\&c.}$

$=a^m+m\frac{a^m x}{a}+\&c.=$ $a^m+ma^{m-1}x+ba^{m-2}x^2+ca^{m-3}x^3+\&c.$

DEMONSTRATION *of the* BINOMIAL THEOREM,

in the caſe of whole poſitive powers.

330. Let the n binomial factors $a+b$, $a'+b'$, $a''+b''$, &c. all conſiſting of different terms, be multiplied together; and their product will, by the nature of multiplication, conſiſt of all the different combinations that can be made of n letters together, by taking one letter from each factor; therefore by art. 319, if $a = a' = a'' =$&c. and $b = b' = b'' =$&c. the product will be equal to

$$a^n + na^{n-1}b + n\,.\,\frac{n-1}{2}a^{n-2}b^2 + \text{\&c. to } \overline{n+1} \text{ terms;}$$

but if $a=a'=a''=$&c. and $b=b'=b''=$&c. the product of the n factors will be equal to $\overline{a+b}\rceil^{n}$; therefore

$$\overline{a+b}\rceil^{n} = a_n + na^{n-1}b + n\,.\,\frac{n-1}{2}\,a^{n-2}b^2$$

$$+ n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,a^{n-3}b^3 + \text{\&c. to } \overline{n+1} \text{ terms.*}$$

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* As it may not be unacceptable to the reader to ſee this important theorem demonſtrated in different ways, the common demonſtration (in the caſe of whole poſitive powers) is here ſubjoined.

Hence, ſince the only difference between $\overline{a+b}\rceil^{n}$ and $\overline{a-b}\rceil^{n}$ is that in the latter, the odd powers of

Let n binomial factors, $x+a$, $x+b$, &c. be multiplied together; and the work will be as follows:

$x+a$
$x+b$

x^2+ax
$\quad +bx+ab$

$x^2+\overline{a+b}\,.\,x+ab$
$x+c$

$x^3+\overline{a+b}\,.\,x^2+abx$
$\quad +cx^2+c\times\overline{a+b}\,.\,x+abc$

$x^3+\overline{a+b+c}\cdot x^2+\overline{ab+ac+bc}\,.\,x+abc$
$x+d$

$x^4+\overline{a+b+c}\,.\,x^3+\overline{ab+ac+bc}\,x^2+abcx$
$\qquad +dx^3+d\times\overline{a+b+c}\,x^2+d\times\overline{ab+ac+bc}\,.\,x+abcd$

$x^4+\overline{a+b+c+d}.x^3+\overline{ab+ac+bc+ad+bd+cd}.x^2+\overline{abc+abd+acd+bcd}.x+abcd$
&c.

And it is evident that whatever n (the number of factors) be, the firſt term of the product will be x^n. And the ſecond term will involve x^{n-1}, whoſe coefficient will be the ſum of *all* the n quantities a, b, &c.—for it will be the ſum of certain of them, and it would be abſurd to exclude one rather than another, ſince whatever were the order in which the n factors were multiplied together, the ſame coefficient muſt reſult. And for a ſimilar reaſon the coefficient of x^{n-2} in the third term will be the ſum of *all* the combinations of the n quantities a, b, &c. two and two together; and the coefficient of x^{n-3} in the fourth term will be the ſum of all their combinations, three and three together; and ſo on, to the laſt or $\overline{n+1}\rceil^{\text{th}}$ term, which will be the product of all the n quantities a, b, c, &c. Hence,

art.

of b will be negative, $\overline{a-b}|^{n} =$

$$a^{n}-na^{n-1}b+n.\frac{n-1}{2}.a^{n-2}b^{2}-\&c. \text{ to } \overline{n+1} \text{ terms.}$$

(art. 311) the coefficient of the ſecond term will conſiſt of n terms; and the coefficient of the third term will conſiſt of $n.\frac{n-1}{2}$ terms; and ſo on. Now let the n quantities $a, b, c,$ &c. be all equal; then ſince any one of them will be equal to a, and the product of any two of them equal to a^{2},—of any three of them equal to a^{3}—and ſo on, the product will become

$$x^{n}+nax^{n-1}+n.\frac{n-1}{2}a^{2}x^{n-2}+\&c. \text{ to } \overline{n+1} \text{ terms.}$$

That the coefficient of the ſecond term of the product of the n factors will be the ſum of *all* the n quantities, a, b, &c.; and the coefficient of the third term, the ſum of all their combinations by pairs; and ſo on; may be more formally demonſtrated thus:

Let the product of n factors be $x^{n}+px^{n-1}qx^{n-2}+rx^{n-3}+\&c.$ to $\overline{n+1}$ terms, where the law is ſuppoſed to obtain. Multiply by a new factor $x+B$, and the new product will be . .
$x^{n+1}+\overline{p+B}.x^{n}+\overline{q+Bp}.x^{n-1}+\overline{r+Bq}.x^{n-2}+\&c.$ to $\overline{n+2}$ terms. But if p be the ſum of the n quantities, $a, b, c,$ &c. $p+B$ will be the ſum of the $\overline{n+1}$ quantities $a, b, c,$ &c. . . . B; and if q be the ſum of all their combinations by pairs, then ſince Bp or $Ba+Bb+Bc+\&c.$ contains all the additional combinations by pairs which reſult from the introduction of B, $q+Bp$ will be the ſum of all the combinations by pairs of the $\overline{n+1}$ quantities $a, b, c,$ &c. B; and ſo on. If therefore this law be found to obtain in one product, it *muſt neceſſarily* obtain in the next ſucceeding, and conſequently muſt obtain univerſally. But it is found to obtain at firſt. Thereſore it obtains univerſally.

GENERAL * DEMONSTRATION of the BINOMIAL THEOREM.

331. Note. *As the following demonſtration depends principally upon the laſt equation in art.* 316, *viz. the equation* $1 + \frac{n-1}{2} + \frac{n-1}{2} \cdot \frac{n-2}{3} + \&c.$ *to* $\overline{n-1}$ *terms* $= \frac{2^n-2}{n}$, *it may ſave unneceſſary trouble to obſerve that, if* 3, 4, 5, &c. *be ſucceſſively ſubſtituted for* n, *this equation will ſucceſſively become*

$$1 + 1 = \frac{2^3-2}{3}; \qquad 1 + \frac{3}{2} + \frac{3}{2} \cdot \frac{2}{3} = \frac{2^4-2}{4};$$

$$1 + \frac{4}{2} + \frac{4}{2} \cdot \frac{3}{3} + \frac{4}{2} \cdot \frac{3}{3} \cdot \frac{2}{4} = \frac{2^5-2}{4}; \ \&c.$$

Let m denote any number whatſoever, integral or fractional, poſitive or negative; and let the coefficients of the 3d, 4th, 5th, 6th, 7th, 8th, 9th, &c. terms of the mth power of any binomial be denoted by b, c, d, e, f, g, h, &c. reſpectively.

And let the coefficients of the ſame terms of the $\overline{m-1}\rceil^{\text{th}}$ power of any binomial be denoted by $\overset{1}{b}$, $\overset{1}{c}$, $\overset{1}{d}$, $\overset{1}{e}$, $\overset{1}{f}$, $\overset{1}{g}$, $\overset{1}{h}$, &c.

And let the coefficients of the ſame terms of the $\overline{m-2}\rceil^{\text{th}}$ power of any binomial be denoted by $\overset{2}{b}$, $\overset{2}{c}$, $\overset{2}{d}$, $\overset{2}{e}$, $\overset{2}{f}$, $\overset{2}{g}$, $\overset{2}{h}$, &c.

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* This demonſtration is independent of the foregoing, and does not require the previous eſtabliſhment of the theorem in the caſe of whole poſitive powers.

And let the coefficients of the ſame terms of the $\overline{m-3}\rceil^{\text{th}}$ power of any binomial be denoted by $\overset{3}{b}$, $\overset{3}{c}$, $\overset{3}{d}$, $\overset{3}{e}$, $\overset{3}{f}$, $\overset{3}{g}$, $\overset{3}{h}$, &c. And ſo on. Where the figures above the letters are not to be conſidered as indices, but merely as diſtinguiſhing marks.

Then ſince, by art. 328, the coefficient of the ſecond term of *any power* of a binomial is the index of that power, it follows that $\overline{1+2x}\rceil^{m} =$ $1 + m.2x + b.2^2x^2 + c.2^3x^3 + d.2^4x^4 + e.2^5x^5 + \&c.$ and

$$\overline{\overline{1+x}+x}\rceil^{m} = \Big(\overline{1+x}\rceil^{m} + m.\overline{1+x}\rceil^{m-1} \times x$$
$$+ b.\overline{1+x}\rceil^{m-2} \times x^2 + c.\overline{1+x}\rceil^{m-3} \times x^3$$
$$+ d.\overline{1+x}\rceil^{m-4} \times x^4 + e.\overline{1+x}\rceil^{m-5} \times x^5 + \&c.$$
$$= 1 + mx + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$
$$+ mx \times \overline{1 + \overline{m-1}.x + \overset{1}{b}x^2 + \overset{1}{c}x^3 + \overset{1}{d}x^4 + \&c.}$$
$$+ bx^2 \times \overline{1 + \overline{m-2}.x + \overset{2}{b}x^2 + \overset{2}{c}x^3 + \overset{2}{d}x^4 + \&c.}$$
$$+ cx^3 \times \overline{1 + \overline{m-3}.x + \overset{3}{b}x^2 + \overset{3}{c}x^3 + \overset{3}{d}x^4 + \&c.}$$
$$+ dx^4 \times \overline{1 + \overline{m-4}.x + \overset{4}{b}x^2 + \overset{4}{c}x^3 + \overset{4}{d}x^4 + \&c.}$$
$$+ ex^5 \times \overline{1 + \overline{m-5}.x + \overset{5}{b}x^2 + \overset{5}{c}x^3 + \overset{5}{d}x^4 + \&c.}$$
$$+ \&c.\Big) =$$

$1 + mx$

$$\begin{array}{r}
1+mx \quad + \quad bx^2 \quad + \quad cx^3 \quad + \quad dx^4 \quad + \quad ex^5+\&c. \\
+mx+m.\overline{m-1}.x^2 \quad + \quad \overset{1}{m}bx^3 \quad + \quad \overset{1}{m}cx^4 \quad + \quad \overset{1}{m}dx^5+\&c. \\
+bx^2+b.\overline{m-2}.x^3 \quad + \quad \overset{2}{b}.bx^4 \quad + \quad \overset{2}{b}cx^5+\&c. \\
+cx^3+c.\overline{m-3}x^4 \quad + \quad \overset{3}{c}bx^5+\&c. \\
+dx^4+d.\overline{m-4}.x^5+\&c. \\
+ex^5+\&c. \\
+\&c.
\end{array}$$

But ſince $1+2x=\overline{1+x}+x$, theſe two ſeries are the ſame in their correſponding parts; i. e. the ſum of the coefficients of any power of x in this laſt ſeries is equal to the coefficient of that ſame power of x in the other ſeries, $1+m.2x+b.2^2x^2+\&c.$

Hence the following equations:

1ſt. $1=1$

2d. $2m=m+m$

3d. $2^2b=b+m.\overline{m-1}+b$

4th. $2^3c=c+b.\overline{m-2}+\overset{1}{m}b+c$

5th. $2^4d=d+c.\overline{m-3}+\overset{2}{b}b+\overset{1}{m}c+d$

6th. $2^5e=e+d.\overline{m-4}+\overset{3}{c}b+\overset{2}{b}c+\overset{1}{m}d+e$

7th. $2^6f=f+e.\overline{m-5}+\overset{4}{d}b+\overset{3}{c}c+\overset{2}{b}d+\overset{1}{m}e+f$

&c.

From the 3d of theſe equations by tranſpoſition 2^2b-2b or $2b=m.\overline{m-1}$.

$$\therefore b=m.\frac{m-1}{2}.$$

Hence the coefficient of the third term of any power of any binomial is known; and the b's are all determined;—viz.

$\overset{1}{b} = \overline{m-1} \, . \, \frac{m-2}{2}$; $\overset{2}{b} = \overline{m-2} \, . \, \frac{m-3}{2}$;

$\overset{3}{b} = \overline{m-3} \, . \, \frac{m-4}{2}$; &c.

Hence, the 4th equation, by tranſpoſition and ſubſtitution, becomes

$$\overline{2^3-2}.c = m.\frac{m-1}{2}.\overline{m-2} + m.\overline{m-1}.\frac{m-2}{2} =$$

$$b \times \overline{m-2} \times \overline{1+1} = b.\overline{m-2} \times \frac{2^3-2}{3} \text{ (art. 316)}$$

$$\therefore c = b \, . \, \frac{m-2}{3}.$$

Hence, the coefficient of the 4th term of any power of any binomial is known; and the c's are all determined;—viz.

$\overset{1}{c} = \overline{m-1} \, . \, \frac{m-2}{2} \, . \, \frac{m-3}{3}$; $\overset{2}{c} = \overline{m-2} \, .$
$\frac{m-3}{2} \, . \, \frac{m-4}{3}$; &c.

Hence, the 5th equation becomes $\overline{2^4-2} \, . \, d$

$$= m \, . \, \frac{m-1}{2} \, . \, \frac{m-2}{3} \, . \, \overline{m-3} \qquad (c \times \overline{m-3})$$

$$+ m \, . \, \frac{m-1}{2} \, . \, \overline{m-2} \, . \, \frac{m-3}{2} \qquad (c \, . \, \overline{m-3} \, . \, \frac{3}{2})$$

$$+ m \, . \, \overline{m-1} \, . \, \frac{m-2}{2} \, . \, \frac{m-3}{3} \qquad (c \, . \, \overline{m-3} \, . \, \frac{3}{2} . \frac{2}{3})$$

$$= c \, . \, \overline{m-3} \times \overline{1 + \frac{3}{2} + \frac{3}{2} . \frac{2}{3}} = c . \overline{m-3} \times \frac{2^4-2}{4}.$$

$$\therefore d = c \, . \, \frac{m-3}{4}.$$

Hence, the coefficient of the 5th term of *any* power of *any* binomial is known, and the *d*'s are all determined—viz.

$$d = \overline{m-1} \,.\, \frac{m-2}{2} \,.\, \frac{m-3}{3} \,.\, \frac{m-4}{4}; \text{ \&c.}$$

Hence, the 6th equation becomes $\overline{2^5 - 2} \,.\, e$

$$= m \,.\, \frac{m-1}{2} \,.\, \frac{m-1}{3} \,.\, \frac{m-3}{4} \,.\, \overline{m-4}$$

$$+ m \,.\, \frac{m-1}{2} \,.\, \frac{m-2}{3} \,.\, \overline{m-3} \,.\, \frac{m-4}{2}$$

$$+ m \,.\, \frac{m-1}{2} \,.\, \overline{m-2} \,.\, \frac{m-3}{2} \,.\, \frac{m-4}{3}$$

$$+ m \,.\, \overline{m-1} \,.\, \frac{m-2}{2} \,.\, \frac{m-3}{3} \,.\, \frac{m-4}{4}$$

$$= d \,.\, \overline{m-4} \times \overline{1 + \frac{4}{2} + \frac{4}{2}.\frac{3}{3} + \frac{4}{2}.\frac{3}{3}.\frac{2}{4}}$$

$$= d \,.\, \overline{m-4} \times \frac{2^5 - 2}{5}$$

$$\therefore e = d \,.\, \frac{m-4}{5}.$$

Hence, the coefficient of the 6th term of any power of any binomial is known, and the *e*'s are all determined.

And the ſame law will be found to obtain univerſally in the ſucceeding coefficients. For let the coefficient of the *n*th term of the ſeries $1 + mx + bx^2 + \text{\&c.}$ be *P*, and the preceding coefficients be *N*, *M*, &c. and the ſucceeding coefficient,

efficient, or coefficient of the $\overline{n+1}\rceil^{th}$ term be $\mathcal{Q}$. Then the $\overline{n+1}\rceil^{th}$ equation will be $2^n\mathcal{Q}=\mathcal{Q}+P.\overline{m-\overline{n-1}}+N\overset{n-2}{b}+M\overset{n-3}{c}\ldots\ldots\ldots$ $\ldots+c\overset{3}{M}+b\overset{2}{N}+m\overset{1}{P}+\mathcal{Q}$; from which, by tranſpoſition, $\overline{2^n-2}.\mathcal{Q}=P.\overline{m-\overline{n-1}}+N\overset{n-2}{b}+M\overset{n-3}{c}$ $\ldots\ldots\ldots+c\overset{3}{M}+b\overset{2}{N}+m\overset{1}{P}$, the right hand ſide of which laſt equation conſiſts of $\overline{n-1}$ * terms. Now if P, N, M, &c. . . . . e, d, c, b have all been found conformable to the law predicted in the binomial theorem, theſe $n-1$ terms are as follows:

$$m.\frac{m-1}{2}\ldots\ldots\frac{m-\overline{n-4}}{n-3}.\frac{m-\overline{n-3}}{n-2}.\frac{m-\overline{n-2}}{n-1}.\overline{m-\overline{n-1}}$$

$$+m.\frac{m-1}{2}\ldots\ldots\frac{m-\overline{n-4}}{n-3}.\frac{m-\overline{n-3}}{n-2}.\overline{m-\overline{n-2}}.\frac{m-\overline{n-1}}{2}$$

$$+m.\frac{m-1}{2}\ldots\ldots\frac{m-\overline{n-4}}{n-3}.\overline{m-\overline{n-3}}.\frac{m-\overline{n-2}}{2}.\frac{m-\overline{n-1}}{3}$$

$$+m.\frac{m-1}{2}\ldots\ldots\overline{m-\overline{n-4}}.\frac{m-\overline{n-3}}{2}.\frac{m-\overline{n-2}}{3}.\frac{m-\overline{n-1}}{4}$$

+ &c.

to

$$+m.\overline{m-1}\ldots\ldots\frac{m-\overline{n-4}}{n-4}.\frac{m-\overline{n-3}}{n-3}.\frac{m-\overline{n-2}}{n-2}.\frac{m-\overline{n-1}}{n-1}$$

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* Theſe terms have two factors each——the left hand factors (beginning from the laſt term) follow this order, $m, b, c, d,$ &c. to P. And the right hand factors (beginning from the firſt term) follow this order, $m-\overline{n-1}$ $\overset{n-2}{b}$, $\overset{n-3}{c}$, $\overset{n-4}{d}$, &c. to $\overset{1}{P}$.

where each term has a common numerator, $m. \overline{m-1} . \overline{m-2} \ldots\ldots . \overline{m-\overline{n-1}}$, and where the difference in the denominators is as follows—viz. The denominator of the ſecond term differs from the denominator of the firſt term in having the factor 2, inſtead of the factor $\overline{n-1}$. The denominator of the third term differs from the denominator of the firſt term, in having the factor 2, inſtead of the factor $\overline{n-1}$, and the factor 3, inſtead of the factor $\overline{n-2}$. The denominator of the fourth term differsfrom the denominator of the firſt term in havingthe factor 2, inſtead of the factor, $\overline{n-1}$, the factor 3, inſtead of the factor $\overline{n-2}$, and the factor 4, inſtead of the factor, $\overline{n+3}$. And ſo on.

Hence, ſince the firſt term is $P . \overline{m-\overline{n-1}}$,

The 2d term $= P . \overline{m-\overline{n-1}} \times \frac{n-1}{2}$

The 3d term $= P . \overline{m-\overline{n-1}} \times \frac{n-1}{2} . \frac{n-2}{3}$

The 4th term $= P . \overline{m-\overline{n-1}} \times \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4}$

And ſo on—Therefore the ſum of the $n-1$ terms is

$$P.\overline{m-\overline{n-1}} \times \overline{1+\frac{n-1}{2}+\frac{n-1}{2}.\frac{n-2}{3}+\&\text{c. to } \overline{n-1} \text{ terms}}$$

$$= P . \overline{m-\overline{n-1}} \times \frac{2^n-2}{n}$$

$$\therefore \overline{2^n-2} \,.\, \mathcal{Q} = P \,.\, \overline{m-\overline{n-1}} \times \frac{2^n-2}{n}$$

$$\therefore \mathcal{Q} = P \,.\, \frac{m-\overline{n-1}}{n}.$$

It appears then, that if the law predicted in the binomial theorem, art. 320, obtain ſo far as *any particular* term, it *muſt neceſſarily* obtain in the *next* ſucceeding term, and conſequently muſt obtain *univerſally*. But it does obtain in the firſt terms—therefore it obtains univerſally. And $\overline{1+2x}\rceil^{m}$

$$= 1 + m \,.\, 2x + m \,.\, \frac{m-1}{2} \times 2^2x^2 + \&c.$$

Hence, $\overline{a+b}\rceil^{m} = a^m \times \overline{1+\frac{b}{a}}\rceil^{m} = a^m \times$

$$\overline{1 + m \,.\, \frac{b}{a} + m \,.\, \frac{m-1}{2} \quad \frac{b^2}{a^2} + \&c.} =$$

$$a^m + ma^{m-1}b + m \,.\, \frac{m-1}{2} \,.\, a^{m-2}b^2 +$$

$m \,.\, \frac{m-1}{2} \,.\, \frac{m-2}{3} \,.\, a^{m-3}b^3 + \&c.$ Which was to be demonſtrated.

EXAMPLES *to the* BINOMIAL THEOREM.

1ſt. $\overline{a+b}\rceil^{8} = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$.

2d. $\overline{a-b}\rceil^{7} = a^{7} - 7a^{6}b + 21a^{5}b^{2} - 35a^{4}b^{3} + 35a^{3}b^{4} - 21a^{2}b^{5} + 7ab^{6} - b^{7}$.

Obſ. As the coefficients in whole poſitive powers follow the ſame order from the laſt term to the firſt, as from the firſt to the laſt, it is unneceſſary in any particular caſe, to compute them further than the middle term.

3d. $\overline{1+1}\rceil^{n}\ (= 2^{n}) = 1 + n + n.\frac{n-1}{2} + n.\frac{n-1}{2}.\frac{n-2}{3} +$ &c.

4th. $\overline{a+b+c}\rceil^{4} = \overline{a+\overline{b+c}}\rceil^{4} = a^{4} + 4a^{3}\times\overline{b+c} + 6a^{2}\times\overline{b+c}\rceil^{2} + 4a\times\overline{b+c}\rceil^{3} + \overline{b+c}\rceil^{4} = a^{4} + 4a^{3}\times\overline{b+c} + 6a^{2}\times\overline{b^{2}+2bc+c^{2}} + 4a\times\overline{b^{3}+3b^{2}c+3bc^{2}+c^{3}} + b^{4} + 4b^{3}c + 6b^{2}c^{2} + 4bc^{3} + c^{4}$

5th. $\overline{1+x}\rceil^{\frac{1}{3}} = 1 + \frac{1}{3}.x + \frac{1}{3}.\frac{\frac{1}{3}-1}{2}.x^{2} +$ &c. $= 1 + \frac{x}{3} - \frac{x^{2}}{9} + \frac{x^{3}}{81} -$ &c. (as in art. 198).

6th. $\overline{a+b}\rceil^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}.a^{\frac{1}{2}-1}b + \frac{1}{2}.\frac{\frac{1}{2}-1}{2}\,a^{\frac{1}{2}-2}b^{2} +$ &c.

$= a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^{2}}{8a^{\frac{3}{2}}} + \frac{b^{3}}{16a^{\frac{5}{2}}} -$ &c. (as in ex. 7. art. 196.)

7th. $\overline{1+x}\rceil^{\frac{m}{n}} = 1 + \frac{m}{n}.x + \frac{m}{n}.\frac{\frac{m}{n}-1}{2}.x^2$

$+ \frac{m}{n}.\frac{\frac{m}{n}-1}{2}.\frac{\frac{m}{n}-2}{3}.x^3 + \&c.$

$= 1 + \frac{m}{n}.x + \frac{m}{n}.\frac{m-n}{2n}.x^2$

$+ \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.x^3 + \&c.$

8th. $\frac{1}{1+x} = \overline{1+x}^{-1} = 1 - 1\times x - 1\times\frac{-1-1}{2}\times x^2$

$- 1\times\frac{-1-1}{2}.\frac{-1-2}{3}\times x^3 + \&c.$

$= 1 - x + x^2 - x^3 + x^4 - \&c.$

9th. $\frac{1}{1-x} = \overline{1-x}^{-1} = 1 - 1\times - x + 1\times x^2$

$- 1\times - x^3 + \&c. = 1 + x + x^2 + x^3 + x^4 + \&c.$
(as in ex. 5th, art. 170, p. 121.)

10th. $\frac{1}{\sqrt{a^2-x^2}} = \overline{a^2-x^2}\rceil^{-\frac{1}{2}} = \overline{a^2}\rceil^{-\frac{1}{2}}$

$- \frac{1}{2}\overline{a^2}\rceil^{-\frac{1}{2}-1}\times - x^2 - \frac{1}{2}.\frac{-\frac{1}{2}-1}{2}.\overline{a^2}\rceil^{-\frac{1}{2}-2}\times x^4 - \&c.$

$= a^{-1} + \frac{1}{2}a^{-3}x^2 + \frac{1}{2}.\frac{3}{2.2}a^{-5}x^4$

$+ \frac{1}{2}.\frac{3}{2.2}.\frac{5}{2.3}a^{-7}x^6 + \&c.$

11th. $\overline{1-x}\rceil^{-2} = 1 - 2 \times -x - 2 . \frac{-2-1}{2} \times x^2$

$- 2 . \frac{-2-1}{2} . \frac{-2-2}{3} \times -x^3 . - \&c.$

$= 1 + 2x + \frac{2 . 3}{2} . x^2 + \frac{2 . 3 . 4}{2 . 3} . x^3$

$+ \frac{2 . 3 . 4 . 5}{2 . 3 . 4} . x^4 + \&c.$

$= 1 + 2x + 3x^2 + 4x^3 + \&c.$

12th. $\frac{a^2+x^2}{a+x} = \overline{a^2+x^2} \times \overline{a+x}\rceil^{-1}$

$= \overline{a^2+x^2} \times \overline{a^{-1} - a^{-2} x + a^{-3} x^2 + \&c.}$

$= a - x + a^{-1} x^2 - a^{-2} x^3 + a^{-3} x^4 - \&c.$

$\qquad + a^{-1} x^2 - a^{-2} x^3 + a^{-3} x^4 - \&c.$

$= a - x + \frac{2x^2}{a} - \frac{2x^3}{a^2} + \&c.$ as in ex. 3d. art. 170.

Cor. Since $\overline{a+b}\rceil^m = a^m + m . a^{m-1} b + m . \frac{m-1}{2} a^{m-2} b^2 + \&c$

And $\overline{a-b}\rceil^m = a^m - m . a^{m-1} b + m . \frac{m-1}{2} a^{m-2} b^2 - \&c$

By addition $\overline{a+b}\rceil^m + \overline{a-b}\rceil^m = 2a^m \; *$

$+ 2m . \frac{m-1}{2} a^{m-2} b^2 \; * + \&c.$

And by ſubtraction $\overline{a+b}\rceil^m - \overline{a-b}\rceil^m = 2ma^{m-1} b \; *$

$+ 2m . \frac{m-1}{2} . \frac{m-2}{3} . a^{m-3} b^3 \; * + \&c.$

SCHOLIUM.

SCHOLIUM.

332. It appears from art. 225, that the *n*th root of $\overline{1+x}\rceil^{r}$ and the *n*th power of $\overline{1+x}\rceil^{\frac{1}{n}}$ can each have but one ſeries. But it does not appear whether or not theſe two ſeries be the *ſame*. Hence, for any thing that can be concluded from that article, $\overline{1+2x}\rceil^{m}$ may, when m is a fraction, whoſe numerator and denominator are both greater than unity, have *two* ſeries. But ſtill it can have *but two;* i. e. $1 + 2mx + 2^{2}bx^{2} +$ &c. and $\overline{1+x}\rceil^{m} + m\,.\,\overline{1+x}\rceil^{m-1} + x + b +$ &c. can *each* have *but two* ſets of values for b, c, d, &c. and thoſe *the ſame in the one ſeries as in the other.* Moreover ſince the coefficient of the ſecond term is *univerſally* the index, there is nothing in the notation uſed in the general demonſtration, which decides whether the ſeries $1 + 2mx + 2^{2}bx^{2} +$ &c. be derived from taking the *n*th root of the *r*th power, or from taking the *r*th power of the *n*th root: neither is there any thing to decide which way the other ſeries, $\overline{1+x}^{m} + m\,.\,\overline{1+x}\rceil^{m-1} \times x +$ &c. is derived. Hence it follows, that the equations obtained by aſſuming the coefficients of the like powers of x to be the ſame in both ſeries, are not reſtricted to give one of the ſets of values for b, c,

&c.

&c. rather than the other, (ſuppoſing there be *two*) but muſt neceſſarily have given both, if there had been two. Since then they give but one, the coefficients b, c, d, &c. have each but one value, or the ſeries that would be obtained by extracting the nth root of the rth power of a binomial, is the ſame as would be obtained by raiſing the nth root of that binomial to the rth power.

FINIS.